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## Generalized modular and orlicz-paranormed sequence spaces and their matrix domains

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### Abstract

We develop a unified modular framework for Orlicz-type sequence spaces generated by either a single Orlicz function or a coordinatewise family of Orlicz functions. Under a uniform  $\Delta_2$  growth hypothesis we prove that the associated Luxemburg functional induces a complete BK-topology. We then introduce Maddox-type variable-exponent Orlicz spaces  $\ell_\psi(p)$  and show that they inherit the same BK-structure. Finally, we study Orlicz-paranormed matrix domains  $\ell_\psi(A)$  under lower triangular matrices with nonzero diagonal, establishing completeness, inclusion criteria driven by global dominance of Orlicz functions and by weight monotonicity, explicit descriptions of the Kothe-Toeplitz  $\alpha$ -,  $\beta$ -, and  $\gamma$ -duals in terms of the conjugate Orlicz function, and concrete Schauder bases transported via  $A^{-1}$ .

**Keywords:** Orlicz sequence spaces, modular sequence spaces, Luxemburg functional,  $\Delta_2$  condition, Maddox-type spaces, variable exponent sequence spaces, BK-spaces, matrix domains, Norlund matrices, Kothe-Toeplitz duals, Schauder basis

### 1. Introduction

Orlicz sequence spaces and their modular generalizations provide a flexible replacement for classical  $\ell_p$  scales, allowing one to model non-power growth, weighted tail control, and variable local behavior in a single functional-analytic setting. The modular approach (via Luxemburg type functionals) is particularly well adapted to matrix transformations and to the study of Kothe-Toeplitz duals; see, for example, Musielak <sup>[2]</sup> and Woo <sup>[1]</sup>. The purpose of this paper is threefold:

1. To present a concise, self-contained BK-space development for modular sequence spaces generated by a family of Orlicz functions under a uniform  $\Delta_2$  condition;
2. To introduce and formalize a Maddox-type Orlicz construction  $\ell_\psi(p)$  (variable exponent in the Orlicz input) and record its basic structural consequences;
3. To transfer these results to Orlicz-paranormed matrix domains  $\ell_\psi(A)$  associated with lower triangular matrices (including Norlund-type matrices), and to derive dual and basis representations in terms of  $\psi^*$  and  $A^{-1}$ .

Throughout, we focus on statements in theorem form and omit repetitive parallel arguments, referring to standard modular-space theory when appropriate.

### 2. Preliminaries: Orlicz functions, modulars, and Luxemburg functionals Orlicz functions and conjugates

**Definition 2.1** (Orlicz function). A function  $\psi: [0, \infty) \rightarrow [0, \infty)$  is an Orlicz function if:

1.  $\psi(0) = 0$  and  $\psi(u) > 0$  for all  $u > 0$ ;
2.  $\psi$  is nondecreasing and convex on  $[0, \infty)$ ;
3.  $\psi(u) \rightarrow \infty$  as  $u \rightarrow \infty$ .

**Definition 2.2** (Conjugate Orlicz function). The conjugate Orlicz function  $\psi^*: [0, \infty) \rightarrow [0, \infty]$  is

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$$\psi^*(v) := \sup_{u \geq 0} uv - \psi(u), \quad v \geq 0$$

A basic tool is Young's inequality (see, e.g., [2]):

$$uv \leq \psi(u) + \psi^*(v) \quad (u, v \geq 0). \quad (1)$$

### Modular sequence spaces

Let  $\omega := \{x = (x_k)_{k \geq 0} : x_k \in \mathbb{C}\}$  denote the space of all complex sequences.

**Definition 2.3** (Family modular). Let  $\Psi = (\psi_k)_{k \geq 0}$  be a sequence of Orlicz functions. For  $x \in \omega$  and  $\mu > 0$  define the modular

$$I_\Psi(x; \mu) := \sum_{k=0}^{\infty} \psi_k(\mu |x_k|) \in [0, \infty], \quad (2)$$

and set  $I_\Psi(x) := I_\Psi(x; 1)$ . The modular sequence space is

$$\ell(\Psi) := \{x \in \omega : I_\Psi(x; \mu) < \infty \text{ for some } \mu > 0\}. \quad (3)$$

**Definition 2.4** (Luxemburg functional). For  $x \in \ell(\Psi)$  define

$$q_\Psi(x) := \inf\{\lambda > 0 : I_\Psi(x/\lambda) \leq 1\}. \quad (4)$$

**Definition 2.5** (Uniform  $\Delta_2$  condition). We say  $\Psi = (\psi_k)$  satisfies a uniform  $\Delta_2$  condition if there exists  $K \geq 1$  such that

$$\psi_k(2u) \leq K \psi_k(u) \quad (u \geq 0, k \geq 0). \quad (5)$$

### Main result I: BK-structure of modular Orlicz sequence spaces

We record the standard BK-space consequence of convex modular theory, specialized to the present family setting; see [1, 2, 3].

**Theorem 3.1** (BK-property of  $\ell(\Psi)$ ). Assume  $\Psi = (\psi_k)$  satisfies the uniform  $\Delta_2$  condition (5).

Then

1.  $q_\Psi$  is a paranorm on  $\ell(\Psi)$  and  $(\ell(\Psi), q_\Psi)$  is complete.
2. Each coordinate functional  $p_k(x) = x_k$  is continuous on  $\ell(\Psi)$ .
3. Consequently,  $q_\Psi$  is equivalent to a norm on  $\ell(\Psi)$ ; in particular,  $\ell(\Psi)$  is a BK-space under this topology.

Proof sketch. Paranorm properties follow from convexity and monotonicity of each  $\psi_k$ , together with the Luxemburg definition (4); the triangle inequality is obtained by convexity after writing  $(x + y)/(\lambda + \mu)$  as a convex combination of  $x/\lambda$  and  $y/\mu$ . Completeness and continuity of coordinates are standard consequences of modular Cauchy criteria combined with (5); see [2, 1]. The final norm-equivalence is obtained by the usual renorming argument for complete K-spaces induced by convex modulars under  $\Delta_2$ ; cf. [2].

### 4. Orlicz and Maddox-type variable-exponent constructions

**Classical Orlicz sequence spaces** Fix a single Orlicz function  $\psi$ .

**Definition 4.1** (Orlicz sequence space). Define

$$I_\psi(x; \mu) := \sum_{k=0}^{\infty} \psi(\mu |x_k|), \quad \ell_\psi := \{x \in \omega : I_\psi(x; \mu) < \infty \text{ for some } \mu > 0\},$$

and the Luxemburg functional

$$q_\psi(x) := \inf\{\lambda > 0 : I_\psi(x/\lambda; 1) \leq 1\}.$$

**Corollary 4.2.** If  $\psi$  satisfies  $\psi(2u) \leq K\psi(u)$  for all  $u \geq 0$ , then  $(\ell_\psi, q_\psi)$  is a BK-space.

Proof. Apply Theorem 3.1 with  $\psi_k \equiv \psi$ .

**Maddox-type Orlicz spaces**  $\ell_\Psi(p)$

**Definition 4.3** (Variable-exponent Orlicz family). Let  $p = (p_k)_{k \geq 0}$  satisfy  $0 < H_1 \leq p_k \leq H_2 < \infty$ . Define  $\psi_k(u) := \psi(u^{p_k})$  and  $\Psi = (\psi_k)_{k \geq 0}$ . Set

$\ell_\Psi(p) := \ell(\Psi)$ ,  
equipped with the Luxemburg functional  $q_\Psi$  from (4). Equivalently,

$$\ell_\psi(p) = \left\{ x \in \omega : \exists \mu > 0 \text{ s.t. } \sum_{k=0}^{\infty} \psi((\mu|x_k|)^{p_k}) < \infty \right\}.$$

**Proposition 4.4** (Inheritance of uniform  $\Delta_2$ ). If  $\psi$  satisfies  $\psi(2u) \leq K\psi(u)$  for all  $u \geq 0$  and  $0 < H_1 \leq p_k \leq H_2 < \infty$ , then  $\Psi = (\psi_k)$  satisfies a uniform  $\Delta_2$  condition (5).

Proof. For  $u \geq 0$ ,

$$\psi_k(2u) = \psi(2u)^{p_k} = \psi(2^{H_2}u)^{p_k} \leq \psi(2^{H_2}u)^{H_2}.$$

Iterating the  $\Delta_2$  inequality finitely many times yields  $\psi(2^{H_2}u) \leq K'\psi(u)$  with  $K'$  depending only on  $K, H_2$ . Hence  $\psi_k(2u) \leq K'\psi_k(u)$  uniformly in  $k$ .

**Corollary 4.5** (BK-property of  $\ell_\psi(p)$ ). Under the hypotheses of Proposition 4.4,  $(\ell_\psi(p), q_\psi)$  is a BK-space.

Proof. Combine Proposition 4.4 with Theorem 3.1.

## 5. Orlicz-paranormed matrix domains

### Definition and completeness

Let  $A = (a_{nk})_{n,k \geq 0}$  be lower triangular with  $a_{nn} \neq 0$ .

**Definition 5.1** (Matrix domain of  $\ell_\psi$ ). Assume  $\psi$  satisfies a  $\Delta_2$  condition. Define

$$\ell_\psi(A) := \{x \in \omega : Ax \in \ell_\psi\}, \quad q_\psi^A(x) := q_\psi(Ax).$$

**Theorem 5.2** (Completeness and BK-property). If  $(\ell_\psi, q_\psi)$  is a BK-space and  $A$  is lower triangular with nonzero diagonal, then  $(\ell_\psi(A), q_\psi^A)$  is a BK-space; in particular it is complete.

Proof. The map  $T_A: \ell_\psi(A) \rightarrow \ell_\psi$ ,  $T_A(x) = Ax$ , is linear and injective. Since  $A$  has nonzero diagonal and is triangular,  $A^{-1}$  exists as a triangular matrix with finite rows, and  $T_A$  is a linear isomorphism onto its range. By definition  $q_\psi^A(x) = q_\psi(T_A x)$ , so  $T_A$  is an isometric embedding.

Completeness and continuity of coordinates transfer from  $\ell_\psi$  to  $\ell_\psi(A)$  via  $T_A$  and  $A^{-1}$ .

### Inclusion principles: dominance and weights

**Definition 5.3** (Global dominance). For Orlicz functions  $\psi_1, \psi_2$ , write  $\psi_2 \leq \psi_1$  if there exist  $a > 0$  and  $C \geq 1$  such that  $\psi_2(u) \leq C\psi_1(au)$  ( $u \geq 0$ ).

**Proposition 5.4** (Inclusion under dominance). Let  $\psi_1, \psi_2$  satisfy  $\Delta_2$  and assume  $\psi_2 \leq \psi_1$ . Then  $\ell_{\psi_1} \subseteq \ell_{\psi_2}$  continuously, and for any triangular  $A$  with nonzero diagonal,  $\ell_{\psi_1}(A) \subseteq \ell_{\psi_2}(A)$  continuously.

Proof. Let  $y \in \ell_{\psi_1}$  so  $\sum_k \psi_1(\lambda|y_k|) < \infty$  for some  $\lambda$ . Dominance gives  $\psi_2((\lambda/a)|y_k|) \leq C\psi_1(\lambda|y_k|)$ ; summing yields  $y \in \ell_{\psi_2}$ . Continuity follows by comparing Luxemburg functionals using dominance and  $\Delta_2$  (standard; see [2]). The matrix-domain inclusion follows by applying the first part to  $Ax$ .

**Definition 5.5** (Weighted Orlicz sequence space). For weights  $w = (w_k)_{k \geq 0}$  with  $w_k > 0$ , define

$$\ell_\psi(w) := \left\{ x \in \omega : \sum_{k=0}^{\infty} \psi(w_k|x_k|) < \infty \right\}.$$

**Lemma 5.6** (Monotonicity in weights). If  $0 < w_k \leq v_k$  for all  $k$ , then  $\ell_\psi(v) \subseteq \ell_\psi(w)$  continuously.

Proof. Since  $\psi$  is nondecreasing,  $\psi(w_k|x_k|) \leq \psi(v_k|x_k|)$  termwise, so the modular for  $w$  is bounded by the modular for  $v$ . Luxemburg functional comparison is immediate.

## 6. Duals and Schauder bases in Orlicz-paranormed matrix domains

### Kothe-Toeplitz duals

For a sequence space  $X \subseteq \omega$  define:

$$X^\alpha := \{y \in \omega : \sum_k |x_k y_k| < \infty \forall x \in X\},$$

$$X^\beta := \{y \in \omega : \sum_k x_k y_k \text{ converges for all } x \in X\},$$

$$X^\gamma := \{y \in \omega : \sup_n \left| \sum_{k=0}^n x_k y_k \right| < \infty \forall x \in X\}.$$

**Proposition 6.1** (Duals of  $\ell_\psi$ ). If  $\psi$  satisfies  $\Delta_2$ , then

$$(\ell_\psi)^\alpha = (\ell_\psi)^\beta = (\ell_\psi)^\gamma = \ell_{\psi^*}.$$

Proof sketch. Young's inequality (1) yields  $\ell_{\psi}^* \subseteq (\ell_{\psi})^{\alpha}$ . Conversely, the classical representation of the Köthe dual of an Orlicz sequence space identifies continuous coordinatewise functionals with  $\ell_{\psi}^*$  under  $\Delta_2$ ; see [2, 4, 5].

**Theorem 6.2** (Duals of  $\ell_{\psi}(A)$ ). Let  $\psi$  satisfy  $\Delta_2$  and let  $A$  be triangular with nonzero diagonal. Let  $B = A^{-1} = (b_{nk})$ . Then

$$(\ell_{\psi}(A))^{\alpha} = (\ell_{\psi}(A))^{\beta} = (\ell_{\psi}(A))^{\gamma} = \{y \in \omega : yB \in \ell_{\psi}^*\},$$

$$\text{where } (yB)_n := \sum_{k=n}^{\infty} y_k b_{kn}.$$

Proof. For matrix domains of BK-spaces, Kothe-Toeplitz duals transform by the inverse matrix:  $X_A^* = \{y : yB \in X^*\}$  for  $\star \in \{\alpha, \beta, \gamma\}$ . Applying this with  $X = \ell_{\psi}$  and using Proposition 6.1 gives the result.

### Schauder bases

**Proposition 6.3** (Canonical basis of  $\ell_{\psi}$ ). If  $\psi$  satisfies  $\Delta_2$ , then the unit vectors  $e^{(n)} = (\delta_{kn})_{k \geq 0}$  form a Schauder basis of  $\ell_{\psi}$ .

Proof sketch. Since  $c_{00}$  is dense in  $\ell_{\psi}$  and coordinate functionals are continuous (Corollary 4.2), the standard coordinate projections  $S_N(x) = \sum_{n=0}^N x_n e^{(n)}$  converge to  $x$  in the Luxemburg topology; see [2].

**Theorem 6.4** (Schauder basis of  $\ell_{\psi}(A)$ ). Let  $\psi$  satisfy  $\Delta_2$  and  $A$  be triangular with nonzero diagonal. Let  $B = A^{-1} = (b_{nk})$  and define  $e_A^{(n)} := (b_{kn})_{k \geq 0}$  (the  $n$ -th column of  $B$ ). Then  $(e_A^{(n)})_{n \geq 0}$  is a Schauder basis of  $\ell_{\psi}(A)$ , and every  $x \in \ell_{\psi}(A)$  has the unique expansion

$$x = \sum_{n=0}^{\infty} (Ax)_n e_A^{(n)} \quad \text{with convergence in } (\ell_{\psi}(A), q_{\psi}^A).$$

Proof. The map  $T_A: \ell_{\psi}(A) \rightarrow \ell_{\psi}$ ,  $T_A(x) = Ax$ , is an isometric isomorphism onto  $\ell_{\psi}$ . Transport the canonical basis of  $\ell_{\psi}$  via  $T_A^{-1} = B$ :  $e_A^{(n)} = B e^{(n)}$  is exactly the  $n$ -th column of  $B$ . Coefficients are forced to be  $(Ax)_n$  by applying  $A$  and using uniqueness in  $\ell_{\psi}$ .

### 7. Concluding remark

The preceding results provide a modular toolkit that simultaneously handles: (i) non-power growth (Orlicz control), (ii) coordinatewise variability (Maddox-type exponents), and (iii) matrix-generated domains relevant in summability and operator theory. In concrete applications (e.g. Nörlund matrices attached to special integer sequences), Theorem 6.2 and Theorem 6.4 reduce dual and basis computations to explicit formulas for  $A^{-1}$ .

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