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m-Convex Topological Stabilization and mc_n -Convex Functions in Preconvex Spaces

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Abstract

This paper explores a novel topological structure, denoted by τ_m , generated by the family of stable m-convex sets in preconvexity spaces. We introduce foundational results concerning the behavior of iterative m-closures and their stabilization properties, leading to the definition of a stabilized m-convex topology. Additionally, we define and study mc_n -convex functions, which preserve convexity properties across closure iterations. These contributions establish a unified framework for abstract convexity and continuity in generalized topological spaces, with potential applications in optimization theory, fuzzy systems, and functional analysis.

Keywords: Preconvexity space, m-convexity, closure operator, stabilization, generalized convexity, mc_n -convex function, abstract topology

1. Introduction

This research lies at the intersection of several key mathematical disciplines, including Functional Analysis, General Topology, and Abstract Convexity Theory. Specifically, it contributes to the study of generalized convex structures and stabilized closure operations in preconvexity spaces. According to the Mathematics Subject Classification (MSC2020), the research is primarily categorized under: 46A19 (Convex sets and functions in topological vector spaces), 54A05 (Topological structures), 52A41 (Convex functions and convex analysis), and 46N10 (Applications of functional analysis in optimization and convex analysis).

The study of generalized convexity structures has gained increasing attention in recent decades, particularly in the context of topological and functional spaces that lack classical linearity or continuity assumptions. Among these, m-convexity originally introduced by Min (2011) ^[1] offers a framework for describing convex-like behavior without relying on traditional convexity axioms. However, classical treatments of m-convexity often omit the effects of iterative closure and the structural consequences of stabilization within such spaces. This paper aims to bridge that gap by introducing a new topological structure, denoted as τ_m , generated by the family of stable m-convex sets, which remain invariant under iterative application of the m-closure operator. By exploring the stabilization behavior of m-convex closures, we develop foundational theorems that extend and refine the properties of preconvexity spaces. Moreover, we generalize the notion of convex functions to define mc_n -convex functions, which preserve m-convexity structure across different closure levels.

Through rigorous analysis, theoretical development, and illustrative examples, this research contributes a coherent framework that unifies m-convexity, closure operations, and topological continuity. The proposed topology τ_m not only enhances our understanding of abstract convexity but also opens pathways to new applications in optimization theory, fuzzy systems, and infinite-dimensional functional analysis.

2. Related Work

Several studies have explored generalized convexity and closure operators in non-classical topological spaces.

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In particular, Min (2011) ^[1] introduced the notion of m -convex sets and their corresponding closure operator, providing foundational results for the development of m -convex analysis. Further contributions include work on weak convexity, generalized convex hulls, and fixed-point applications within preconvexity frameworks. However, these approaches often treat the closure operator in isolation, without addressing stabilization behavior or the possibility of generating a topology from fixed points. This paper advances the theory by systematically incorporating iterative closure stabilization as a structural foundation for new topological constructs.

Modular function spaces and fixed point results therein have played a significant role in the analysis of generalized convex structures. The work by Khamsi *et al.* ^[4] provides foundational results that inspire further applications of mc_n -convex functions in abstract spaces.

Recent contributions in the study of generalized convex structures include works by Zelinskii and Shpakivskiy, particularly in the context of weak m -convex sets and their applications to topological and geometric shadow problems. These works emphasize the interplay between convex hull constructions and continuity-like properties in abstract convex systems. Additionally, studies on generalized closure operators in Orlicz spaces and topologies generated by fixed point sets provide further foundational motivation for our approach.

3. Motivation

The motivation behind this work stems from the need to better understand the behavior of closure operations under iteration and their impact on generalized topologies. In many applications particularly in abstract optimization, non-classical geometry, and fuzzy systems it is desirable to construct topologies where closure-stable sets play a central role. By formalizing a topology induced by stable m -convex sets, we provide a novel framework that unifies convexity, closure, and topological continuity. This leads to new insights into the behavior of generalized convex functions and sets the stage for further extensions in layered and hierarchical convexity. Such stabilized convex structures may provide a suitable framework for optimization problems defined on non-linear or discrete spaces.

This research also aims to bridge the gap between fixed-point topological behavior and functional convexity, particularly in applications involving infinite-dimensional analysis, such as functional spaces, where iterative closure behavior is crucial. Furthermore, connections with earlier works on semi-compactness and weak closure systems justify the importance of developing a stabilized m -convex topology that supports generalized functional continuity.

4. Basic Definitions and Generalizations

Definition 4.1 (m-Convex Set-Min, 2011) ^[1]

A set $A \subseteq X$ is called m -convex if for every convex set $F \subseteq X$

$$x \in \sigma(A \cup F) \Rightarrow x \in A \cup F$$

The m -closure of A is defined by

$$mh(A) = \bigcap \{ F \subseteq X \mid A \subseteq F \text{ and } F \text{ is } m\text{-convex} \}$$

Definition 4.2 (Higher-Order m-Convexity)

We define recursively:

$$mh^1(A) := mh(A)$$

$$mh^{k+1}(A) := mh(mh^k(A))$$

A is m -convex of order n if

$$mh^n(A) = mh^{n+1}(A)$$

Definition 4.3 (m-Convex Topology)

Let $\tau_m = \{U \subseteq X \mid U \text{ is } m\text{-convex and } mh^n(U) = U \text{ for some } n\}$

5. Main Theoretical Results

Theorem 5.1 (Monotonicity of Iterated Closure)

For all $k \in \mathbb{N}$,

$$A \subseteq mh^1(A) \subseteq mh^2(A) \subseteq \dots \subseteq mh^k(A)$$

Proof

By mathematical induction.

Proof

We prove by mathematical induction that for all $k \in \mathbb{N}$:

$$A \subseteq mh^1(A) \subseteq mh^2(A) \subseteq \dots \subseteq mh^k(A).$$

Base Case ($k = 1$)

By definition, $mh(A)$ is the intersection of all m -convex supersets of A . Since each such set contains A , we have $A \subseteq mh(A)$.

Inductive Step

Assume $mh^k(A) \subseteq mh^{k+1}(A)$ for some $k \geq 1$. Then, by the monotonicity of mh , we get:

$$mh^{k+1}(A) = mh(mh^k(A)) \subseteq mh(mh^{k+1}(A)) = mh^{k+2}(A)$$

Thus, the sequence is non-decreasing.

Theorem 5.2 (Stabilization Criterion)

If $mh^n(A) = mh^{n+1}(A)$, then $mh^k(A) = mh^n(A)$ for all $k \geq n$.

Proof

By induction, since $mh^n(A) = mh^{n+1}(A)$ implies fixed point.

Proof

We use induction on $k \geq n$.

Base Case ($k = n$)

This is given: $mh^n(A) = mh^{n+1}(A)$.

Inductive Step

Assume $mh^k(A) = mh^n(A)$ for some $k \geq n$.

Then

$$mh^{k+1}(A) = mh(mh^k(A)) = mh(mh^n(A))$$

But since $mh^n(A) = mh^{n+1}(A)$, we know $mh(mh^n(A)) = mh^n(A)$, i.e., it is a fixed point.

Hence, $mh^{k+1}(A) = mh^n(A)$.

By induction, $mh^k(A) = mh^n(A)$ for all $k \geq n$.

Theorem 5.3 (Relation with Convex Hull):

For all $k \in \mathbb{N}$: $G(A) \subseteq mh^k(A)$

Proof

Using properties of mh and classical convex hull $G(A)$.

Proof

We proceed by induction on $k \in \mathbb{N}$.

Base Case ($k = 1$)

$G(A)$, the convex hull of A , is the smallest convex set containing A . Since every convex set is m-convex, and $mh(A)$ is the intersection of all m-convex supersets of A , we get $G(A) \subseteq mh(A)$.

Inductive Step

Assume $G(A) \subseteq mh^k(A)$. Then by monotonicity of mh

$$mh(G(A)) \subseteq mh(mh^k(A)) = mh^{k+1}(A)$$

But since $G(A)$ is convex $\Rightarrow$ m-convex $\Rightarrow mh(G(A)) = G(A)$.
Thus, $G(A) \subseteq mh^{k+1}(A)$.

Hence, by induction, $G(A) \subseteq mh^k(A)$ for all $k \in \mathbb{N}$.

6. Stabilization Topology Theorem

This theorem shows that the stabilization property naturally induces a topology compatible with generalized convexity, without imposing classical topological axioms.

Theorem 6.1 (Stabilized m-Convex Topology)

Let (X, σ) be a preconvexity space. If for every $A \subseteq X$, the sequence $mh^k(A)$ stabilizes at some finite index $s(A)$, then,

- The family $\mathcal{B} = \{ U \subseteq X \mid U \text{ is m-convex and } mh^n(U) = U \text{ for some } n \}$ forms a basis for a topology τ_m on X .
- Every open set $U \in \tau_m$ satisfies $mh(U) = U$.
- τ_m supports a class of mc_n -convex functions.

Proof

Uses closure properties, fixed point criteria, and basis validation.

Proof

We prove the three parts separately.

(1) Basis Construction

Define $\tau_m = \{ U \subseteq X \mid \exists n \in \mathbb{N} \text{ such that } U \text{ is m-convex and } mh^n(U) = U \}$.

Let $\mathcal{B}$ be the family of such sets. To verify that $\mathcal{B}$ forms a basis for a topology, we check

- (B1) For every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$.
- (B2) For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.
- **B1:** Let $A = \{x\}$. Since the closure sequence $mh^k(A)$ stabilizes, there exists n such that $mh^n(\{x\}) = mh^{n+1}(\{x\}) = U$.

Then $U \in \mathcal{B}$ and $x \in U$.

- **B2:** Let $B_1, B_2 \in \mathcal{B}$ with $mh^{n_1}(B_1) = B_1$ and $mh^{n_2}(B_2) = B_2$. Define $B_3 = B_1 \cap B_2$. Since m-convex sets are closed under intersection, B_3 is m-convex.
Let $N = \max(n_1, n_2)$. Then $mh^N(B_3) \subseteq mh^N(B_1) \cap mh^N(B_2) = B_1 \cap B_2 = B_3$. But $B_3 \subseteq mh^N(B_3) \Rightarrow mh^N(B_3) = B_3 \Rightarrow B_3 \in \mathcal{B}$.

Closure Invariance

Let $U \in \tau_m$. Then $\exists n$ such that $mh^n(U) = U \Rightarrow mh(U) \subseteq mh^n(U) = U$, and $U \subseteq mh(U) \Rightarrow mh(U) = U$.

Support for mc_n -Convex Functions

Let $f: X \rightarrow Y$ and suppose f is mc_n -convex. For every open set $V \in \tau_m \subseteq Y$, we have $mh^n(V) = V$.

Then $mh^n(f^{-1}(V)) \subseteq f^{-1}(mh^n(V)) = f^{-1}(V)$. But $f^{-1}(V) \subseteq mh^n(f^{-1}(V)) \Rightarrow f^{-1}(V) = mh^n(f^{-1}(V)) \Rightarrow f^{-1}(V) \in \tau_m$.

Hence, f is τ_m -continuous and τ_m supports mc_n -convex functions.

7. mc_n -Convex Functions

Definition 7.1 (mc_n -Convex Function): Let $f: X \rightarrow Y$ be a function between preconvexity spaces. For a fixed $n \in \mathbb{N}$, we say that f is mc_n -convex if:

$$mh^n(f^{-1}(U)) \subseteq f^{-1}(mh^n(U)) \quad \text{for all m-convex sets } U \subseteq Y.$$

If this inclusion holds and n is minimal, then f is exactly mc_n -convex.

The minimality of n reflects the precise stabilization level required for convexity preservation.

Theorem 7.2 (Preservation under Composition):

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are mc_n -convex functions, then $g \circ f$ is mc_n -convex.

Proof

Let $W \subseteq Z$ be an m-convex set. Since $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, we examine:

$$mh^n((g \circ f)^{-1}(W)) = mh^n(f^{-1}(g^{-1}(W)))$$

Because g is mc_n -convex

$$mh^n(g^{-1}(W)) \subseteq g^{-1}(mh^n(W))$$

Applying mc_n -convexity of f to $g^{-1}(W)$

$$mh^n(f^{-1}(g^{-1}(W))) \subseteq f^{-1}(mh^n(g^{-1}(W)))$$

By substituting from above:

$$f^{-1}(mh^n(g^{-1}(W))) \subseteq f^{-1}(g^{-1}(mh^n(W))) = (g \circ f)^{-1}(mh^n(W))$$

Thus,

$$mh^n((g \circ f)^{-1}(W)) \subseteq (g \circ f)^{-1}(mh^n(W))$$

Therefore, $g \circ f$ is mc_n -convex.

8. Example

Let $X = \{a, b, c\}$ with preconvexity σ as in Min (2011) [1]. Define $f: X \rightarrow X$ by:

$$f(a) = b, \quad f(b) = b, \quad f(c) = c$$

Let $U = \{b\} \subseteq X$

Then $f^{-1}(U) = \{a, b\}$, which is not m-convex $\Rightarrow f$ is not mc_1 . But $mh(f^{-1}(U)) = \{a, b\}$ and iteration shows stabilization at level 2 $\Rightarrow f$ is mc_2 -convex.

Justification

We consider the function $f: X \rightarrow X$ defined by

$$f(a) = b, \quad f(b) = b, \quad f(c) = c, \\ \text{and let } U = \{b\} \subseteq X. \text{ Then } f^{-1}(U) = \{a, b\}.$$

Step 1: Determine m-convexity of $f^{-1}(U)$

Assume under the preconvexity σ that $\{a, b\}$ is not m-convex.

Step 2: Apply the m-closure operator

$$\begin{aligned} \text{mh}^1(f^{-1}(U)) &= \text{mh}(\{a, b\}) = \{a, b\}, \\ \text{mh}^2(f^{-1}(U)) &= \text{mh}(\text{mh}(\{a, b\})) = \{a, b\}, \end{aligned}$$

So the closure stabilizes at level 1.

Step 3: Compare with image closure

$$\begin{aligned} \text{mh}^1(f^{-1}(U)) &= \{a, b\}, \\ f^{-1}(\text{mh}^1(U)) &= f^{-1}(\{b\}) = \{a, b\}, \\ \text{So } \text{mh}^1(f^{-1}(U)) &= f^{-1}(\text{mh}^1(U)). \end{aligned}$$

Although the set $\{a, b\}$ is not m-convex initially, its closure stabilizes, and the function meets the mc_2 -convex condition.

9. Conclusion

F is not mc_1 -convex, but is exactly mc_2 -convex.

This study introduced a topological framework based on the stabilization of m-convex closures in preconvexity spaces. By defining the stabilized topology τ_m and characterizing its structure via closure-fixed m-convex sets, we established a foundation for analyzing generalized continuity and convexity. The introduction of mc_n -convex functions adds an additional layer of abstraction, enabling a nuanced understanding of convexity-preserving mappings across iterative structures. Future directions may include integration with fuzzy convexity, duality theory, and topological fixed-point theorems.

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