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Multi-Fuzzy linear spaces over multi-fuzzy field with respect to a bridge function

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Abstract

The concept of multi-fuzzy vector space over multi-fuzzy field generalizes Nanda's concept of fuzzy vector space over fuzzy field. This paper explores the properties of multi-fuzzy vector space, denoted by $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of the vector space V over the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of the field F , assuming that the dimensions of μ and λ are equal. Additionally, we introduce a new class of multi-fuzzy vector spaces using bridge functions. When the dimensions of the multi-fuzzy set μ of V and the multi-fuzzy field λ differ, a bridge function h is employed to define the multi-fuzzy vector space. Furthermore, this paper examines some algebraic structure of these spaces and investigates how the bridge function h affects their properties.

Keywords: Multi-fuzzy set, multi-fuzzy field, multi-fuzzy vector spaces, multi-fuzzy vector spaces with respect to a bridge function

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965. Rosenfeld extended fuzzy set theory by defining fuzzy groupoids and fuzzy groups, which paved the way for further advancements such as fuzzy rings and fuzzy fields. The foundational concepts of fuzzy vector spaces were later formalized by Katsara and Liu, inspiring significant research and new theoretical developments in this area. Nanda advanced the field by introducing fuzzy vector spaces over fuzzy field, generalizing the notion of fuzzy vector spaces over conventional field. In 2010, Sabu Sebastian and T.V. Ramakrishnan introduced the concept of multi-fuzzy sets and investigated the properties of multi-fuzzy groups, broadening the scope of fuzzy mathematics. Inspired by these developments, this paper explores the properties of multi-fuzzy vector spaces μ of V over the multi-fuzzy field λ of the field F , assuming that the dimensions of μ and λ are equal.

In addition, we introduce a new class of multi-fuzzy vector spaces using bridge functions, say h . When the dimensions of the multi-fuzzy set μ and the multi-fuzzy field λ differ, the bridge function h plays a crucial role in defining these spaces. The primary focus of this paper is to study into the algebraic structure of these spaces and analyze the influence of the bridge function on their properties, offering new insights into multi-fuzzy linear algebra.

2. Preliminaries

Definition 2.1. Let $\{\langle L_i, \vee_i, \wedge_i \rangle, 1 \leq i \leq k\}$ be a family of complete distributive lattices. Then the product $L = \prod_{i=1}^k L_i$ form a lattice with *supremum* $\vee$ and *infimum* $\wedge$ are defined by

$$(a_1, a_2, \dots, a_k) \vee (b_1, b_2, \dots, b_k) = (a_1 \vee_1 b_1, a_2 \vee_2 b_2, \dots, a_k \vee_k b_k)$$

$$(a_1, a_2, \dots, a_k) \wedge (b_1, b_2, \dots, b_k) = (a_1 \wedge_1 b_1, a_2 \wedge_2 b_2, \dots, a_k \wedge_k b_k).$$

Remark: In all cases we consider each lattice to be a complete distributive lattice. The supremum and infimum elements of the lattice L are denoted by 1_L and 0_L respectively.

Definition 2.2 [3]. Let $L = \prod_{j=1}^k L_j$ be a product complete distributive lattices. Let $t_j: L_j \times L_j \rightarrow L_j$ for $j = 1, 2, \dots, k$ be t-norms. The T-operator (or T-norm) $T: L \times L \rightarrow L$ is defined by

$$T(x, y) = (t_1(x_1, y_1), t_2(x_2, y_2), \dots, t_k(x_k, y_k))$$

Where $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k) \in L$

Let $s_j: L_j \times L_j \rightarrow L_j$ for $j = 1, 2, \dots, k$ be t-co-norms

The S-operator $S: L \times L \rightarrow L$ is defined by

$$S(x, y) = (s_1(x_1, y_1), s_2(x_2, y_2), \dots, s_k(x_k, y_k))$$

We denote $T = (t_1, t_2, \dots, t_k)$ and $S = (s_1, s_2, \dots, s_k)$.

Definition 2.3[3]: Let X be a set, and let $L = \prod_{j=1}^k L_j$ be a product of lattices. Let k be a positive integer. A multi-fuzzy set A in X is a set of ordered $k+1$ -tuples

$A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)): x \in X\}$, where $\mu_j \in L_j^X$, $1 \leq j \leq k$. The function $\mu = (\mu_1, \mu_2, \dots, \mu_k): X \rightarrow L$ is called multi-fuzzy membership function and k is called the dimension of A (or the dimension of μ). The set of all multi-fuzzy sets of X to L is denoted by $M^kFS(X, L)$.

3. Multi fuzzy vector space over multi fuzzy field under T-norm

Definition 3.1. Let $L = \prod_{j=1}^k L_j$ be a product complete distributive lattices, and let $T = (t_1, t_2, \dots, t_k)$ be a T-norm on L . Let $F(+, \cdot)$ be a field. A multi fuzzy set $A = \{(p, \lambda_1(p), \lambda_2(p), \dots, \lambda_k(p)): p \in F\}$, where $\lambda_j \in L_j^F$, $1 \leq j \leq k$, with membership function $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is called a multi fuzzy field of F under T-norm T if it satisfies, for all $p, q \in F$:

$$a) \lambda(p + q) \geq T(\lambda(p), \lambda(q)) = (t_1(\lambda_1(p), \lambda_1(q)), t_2(\lambda_2(p), \lambda_2(q)), \dots, t_k(\lambda_k(p), \lambda_k(q))).$$

$$b) \lambda(-p) \geq \lambda(p), \text{ for all } p \text{ in } F.$$

$$c) \lambda(p \cdot q) \geq T(\lambda(p), \lambda(q)) = (t_1(\lambda_1(p), \lambda_1(q)), t_2(\lambda_2(p), \lambda_2(q)), \dots, t_k(\lambda_k(p), \lambda_k(q))).$$

$$d) \lambda(p^{-1}) \geq \lambda(p), \text{ for all non zero element } p \in F.$$

$$e) \lambda(0_F) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k}) = \lambda(1_F),$$

Where 0_F is the zero element and 1_F is the unity in F . 1_{L_j} is the supremum of L_j for each $1 \leq j \leq k$ so that 1_L is supremum of L .

The set all multi fuzzy field of F under T-norm $T = (t_1, t_2, \dots, t_k)$ is denoted by $MFF^k(F, L, T)$

We say that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is a multi-fuzzy field of F and write $\lambda \in MFF^k(F, L, T)$.

If T is the minimum t-norm, then $MFF^k(F, L, T)$ is denoted by $MFF^k(F, L)$

Definition 3.2: Let $L = \prod_{j=1}^k L_j$ be a product of lattices, and let $T = (t_1, t_2, \dots, t_k)$ be a T-norm on L . Let V be a vector space over the field F , and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k) \in MFF^k(F, L, T)$. A multi-fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)): x \in V\}$, where $\mu_j \in L_j^V$ for $1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, is called a multi-fuzzy vector space over the multi-fuzzy field λ of F under T-norm T if it satisfies the following conditions for all $x, y \in V$ and $p \in F$:

$$a) \mu(x + y) \geq T(\mu(x), \mu(y)) = (t_1(\mu_1(x), \mu_1(y)), t_2(\mu_2(x), \mu_2(y)), \dots, t_k(\mu_k(x), \mu_k(y)))$$

$$b) \mu(-x) \geq \mu(x), \text{ for all } x \in V$$

$$c) \mu(p \cdot x) \geq T(\lambda(p), \mu(x)) = (t_1(\lambda_1(p), \mu_1(x)), t_2(\lambda_2(p), \mu_2(x)), \dots, t_k(\lambda_k(p), \mu_k(x)))$$

$$d) \mu(0_V) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k}), \text{ where } 0_V \text{ is the zero vector in } V \text{ and } 1_L \text{ is the largest element in } L$$

The set all multi-fuzzy vector space of V over the multi-fuzzy field λ of F under the T-norm T is denoted by $MFV^k(V, \lambda, L, T)$. If the T-norm is the minimum, then $MFV^k(V, \lambda, L, T)$ is denoted by $MFV^k(V, \lambda, L)$.

We say that $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ is a multi-fuzzy vector space of V over the multi-fuzzy Field λ of F and denote it as $\mu \in MFV^k(V, \lambda, L, T)$.

Remark 3.1: If we consider $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ as a multi-fuzzy vector space over the ordinary field F instead of the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of F , where $\lambda(p) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k})$, then condition (c) reduces to:

$$\mu(p.x) \geq T(\lambda(p), \mu(x)) = T(1_L, \mu(x)) = \mu(x).$$

All other conditions remain the same.

The set of all multi-fuzzy vector space of V over the ordinary field F under T -norm $T = (t_1, t_2, \dots, t_k)$ is denoted by $MFV^k(V, L, T)$.

Theorem 3.1. Let V be a vector space over the field F , and let μ be a multi-fuzzy subset of V with

$$\mu(0_V) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k}). \text{ Then:}$$

1. $\mu \in MFV^k(V, \lambda, L, T)$ if and only if $\mu(px + qy) \geq T(T(\lambda(p), \mu(x)), T(\lambda(q), \mu(y)))$ for all $x, y \in V$ and for all $q, p \in F \dots (A)$.
2. $\mu \in MFV^k(V, L, T)$ if and only if $\mu(px + qy) \geq T(\mu(x), \mu(y))$ for all $x, y \in V$ and for all $q, p \in F$

Definition 3.3(Multi-fuzzy vector space based on lattice operations)

Let V be a vector space over the field F , and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a multi-fuzzy field of F . A multi-fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)) : x \in V\}$, where $\mu_j \in L_j^V, 1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, is called a multi-fuzzy vector space over the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of F if it satisfies the following conditions for all $x, y \in V$ and $p \in F$

- a). $\mu(x + y) \geq \mu(x) \wedge \mu(y) = (\mu_1(x) \wedge \mu_1(y), \mu_2(x) \wedge \mu_2(y), \dots, \mu_k(x) \wedge \mu_k(y))$
- b). $\mu(-x) \geq \mu(x)$, for all $x \in V$.
- c). $\mu(p.x) \geq \lambda(p) \wedge \mu(x) = ((\lambda_1(p) \wedge \mu_1(x)), (\lambda_2(p) \wedge \mu_2(x)), \dots, (\lambda_k(p) \wedge \mu_k(x)))$
- d). $\mu(0_V) = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k}) = 1_L$, the greatest member of L .

The set of all multi-fuzzy vector space of V over the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of F , based on lattice operations of $L = \prod_{j=1}^k L_j$ is denoted by $MFV^k(V, \lambda, L)$.

We say that $\mu = (\mu_1, \mu_2, \dots, \mu_k)$, is a multi-fuzzy vector space of V over the multi fuzzy field λ of F and denote it as $\mu \in MFV^k(V, \lambda, L)$.

Theorem 3.2. Let V be a vector space over field F , and let W be a subset of V . Let $\mu: V \rightarrow L = \prod_{j=1}^k L_j$ be defined by: $\mu(x) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k})$, if $x \in W$ and $\mu(x) = 0_L$ if $x \notin W$. Then, $\mu \in MFV^k(V, \lambda, L)$ if and only if W is a subspace of V .

Theorem 3.3. Let V be a vector space over field F , and W be a subset of V . Let $\mu \in MFV^k(V, \lambda, L)$, Then $W = \{x \in V : \mu(x) = 1_L\}$ is a subspace of V .

Proof. Let $x, y \in W$ and $p \in F$, Then $\mu(px + y) \geq \inf\{\mu(x), \mu(y)\} = \inf\{1_L, 1_L\} = 1_L$.

Since $\mu(px + y) \leq 1_L$, this implies $\mu(px + y) = 1_L$. Therefore, $px + y \in W$, showing that $W = \{x \in V : \mu(x) = 1_L\}$ is a subspace of V .

Theorem 3.4. Let $\{\mu_j \in MFV^k(V, \lambda, L) : j \in J\}$ be an indexed family of multi-fuzzy vector space of V over the multi-fuzzy field λ of F . Then their intersection $\mu = \bigcap_{j \in J} \mu_j \in MFV^k(V, \lambda, L)$

Theorem 3.5. Let V be a vector space over the field F . Let μ and τ be in $MFV^k(V, \lambda, L)$.

Then the multi-fuzzy set $\mu \times \tau : V \times V \rightarrow L$ is defined by

$(\mu \times \tau)(z_1, z_2) = \inf\{\mu(z_1), \tau(z_2)\}$ for all $(z_1, z_2) \in V \times V$, and is a multi-fuzzy vector space of $V \times V$ over the multi-fuzzy field λ of F . That is,

$$\mu \times \tau \in MFV^k(V \times V, \lambda, L).$$

4. Multi-fuzzy vector space μ over multi-fuzzy field λ with $\dim \mu \neq \dim \lambda$

Definition 4.1. Let $M = \prod_{i=1}^m M_i$ and $L = \prod_{j=1}^k L_j$ be products of lattices. Let A be the multi-fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)) : x \in X\}$, where $\mu_j \in L_j^X$ for $1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ defined on a set X with value domain $L = \prod_{j=1}^k L_j$. Let $h: L \rightarrow M$ be a crisp function.

The multi-fuzzy set B of X is defined by:

$$B = \{(x, \mu^h(x)) : x \in X\}, \text{ where the membership function } \mu^h: X \rightarrow \prod_{i=1}^m M_i \text{ is defined by:}$$

$\mu^h(x) = h(\mu(x))$ for all x .

The function μ^h is called fuzzy transformation of multi fuzzy set μ with respect to the crisp function h . Clearly, $\mu \in M^k FS(X, L)$ and $\mu^h \in M^m FS(X, M)$.

In the above definition, we can replace $M = \prod_{i=1}^m M_i$ with any lattice. For example lattice of matrices

Example 4.1. Let A is the multi fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)) : x \in V\}$, where $\mu_j \in I^V$ for $1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ defined on a set V with value domain $L = I^k$, which forms a lattice, where $I = [0, 1]$. Let $B = [b_{ij}]$ is a $k \times m$ fuzzy matrix on I . Let $h: L \rightarrow M$ be the crisp function defined by:

$h(a_1, a_2, a_3, \dots, a_k) = [a_1 \ a_2 \ a_3 \ \dots \ a_k] [b_{ij}]$, where the expression denotes the fuzzy product of matrices.

The range of h is a subset of the set of all $1 \times m$ matrices over $I = [0, 1]$.

$\mu^h(x) = h(\mu(x)) = h(\mu_1(x), \mu_2(x), \dots, \mu_k(x)) = [\mu_1(x) \ \mu_2(x) \ \dots \ \mu_k(x)] [b_{ij}]$.

Clearly, μ^h is a multi-fuzzy subset of V to a subset of the set of all $1 \times m$ matrices over $I = [0, 1]$.

Example 4.2. Let A is the multi fuzzy set $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)) : x \in X\}$, where $\mu_j \in I^X$, $1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ defined on a set X with value domain $L = I^k$, which forms a lattice, where $I = [0, 1]$. Let $B = [b_{ij}]$ be a $1 \times m$ fuzzy matrix on I . Let $h: L \rightarrow M$ be the crisp function given by:

$h(a_1, a_2, a_3, \dots, a_k) = [a_1 \ a_2 \ a_3 \ \dots \ a_k]^t [b_{ij}]$, where the expression denotes the fuzzy product of matrices, and $[a_1 \ a_2 \ a_3 \ \dots \ a_k]^t$ represents the transpose of the matrix $[a_1 \ a_2 \ a_3 \ \dots \ a_k]$.

The range of h is a subset of the set of all $k \times m$ matrices over $I = [0, 1]$.

$\mu^h(x) = h(\mu(x)) = h(\mu_1(x), \mu_2(x), \dots, \mu_k(x)) = [\mu_1(x) \ \mu_2(x) \ \dots \ \mu_k(x)] [b_{ij}]$

Clearly, μ^h is a multi-fuzzy subset of X to a subset of the set of all $k \times m$ matrices over $I = [0, 1]$, which is a new type of multi-fuzzy subset.

Theorem 4.1. Let $\mu, \tau \in M^k FS(X, L)$ and $h: L \rightarrow M$ be a monotonic function. Then:

- If h increasing and $\mu \subseteq \tau$, then $\mu^h \subseteq \tau^h$.
- If h is decreasing and $\mu \subseteq \tau$, then $\tau^h \subseteq \mu^h$.
- If h is increasing, then $(\mu \cap \tau)^h \subseteq \mu^h \cap \tau^h$.
- If h is increasing, then $\mu^h \cup \tau^h \subseteq (\mu \cup \tau)^h$.
- In general, $(\mu \cap \tau)^h \neq \mu^h \cap \tau^h$ and $\mu^h \cup \tau^h \neq (\mu \cup \tau)^h$.

Proof: a) Let $\mu, \tau \in M^k FS(X, L)$ and $\mu \subseteq \tau$, so $\mu(x) \leq \tau(x)$, for all $x \in X$

If h is increasing, then

$\mu^h(x) = h(\mu(x)) \leq h(\tau(x)) = \tau^h(x)$, for all $x \in X$

Thus, $\mu^h \subseteq \tau^h$

- Let h is decreasing and $\mu \subseteq \tau$, then $\mu(x) \leq \tau(x)$, for all $x \in X$ and $\mu^h(x) = h(\mu(x)) \geq h(\tau(x)) = \tau^h(x)$, for all $x \in X$.

Thus, $\tau^h \subseteq \mu^h$

- Let h is increasing. we have, $\mu \cap \tau \subseteq \mu, \tau$

By using the first result we have, $(\mu \cap \tau)^h \subseteq \mu^h$ and $(\mu \cap \tau)^h \subseteq \tau^h$

Therefore, $(\mu \cap \tau)^h \subseteq \mu^h \cap \tau^h$

- Let h is increasing. We have, $\mu, \tau \subseteq \mu \cup \tau$

By using the first result we have, $\mu^h \subseteq (\mu \cup \tau)^h$ and $\tau^h \subseteq (\mu \cup \tau)^h$

Therefore, $\mu^h \cup \tau^h \subseteq (\mu \cup \tau)^h$.

Next, we introduce another type of multi-fuzzy vector space using a bridge function. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a multi-fuzzy field of F , and let $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ be a multi-fuzzy set in V of dimension $k \neq m$. In this case, a direct definition of the multi-fuzzy vector space over λ is not feasible. To address this, a bridge function $h: M = \prod_{i=1}^m M_i \rightarrow L = \prod_{j=1}^k L_j$ is introduced.

Definition 4.2.(Multi-fuzzy vector space μ over multi fuzzy field λ with $\dim \mu \neq \dim \lambda$)

Let $M = \prod_{i=1}^m M_i$ and, $L = \prod_{j=1}^k L_j$ be products of complete distributive lattices. Let V be a vector space over the field F , and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in MFF^m(F, M, U)$. Let A be the multi fuzzy set defined by $A = \{(x, \mu_1(x), \mu_2(x), \dots, \mu_k(x)) : x \in V\}$, where $\mu_j \in L_j^V$, $1 \leq j \leq k$, with membership function $\mu = (\mu_1, \mu_2, \dots, \mu_k)$. If $k \neq m$ and let $h: M \rightarrow L$ is a crisp function, then μ is called a multi-fuzzy vector space over the multi fuzzy field λ of F under T-norm T (defined on L) with respect to the function h , if for all $x, y \in V$ and $p \in F$ the following conditions hold:

- $\mu(x + y) \geq T(\mu(x), \mu(y))$
- $\mu(-x) \geq \mu(x)$, For all $x \in V$
- $\mu(p \cdot x) \geq T(\lambda^h(p), \mu(x))$
- $\mu(0_V) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k})$,

Where 0_V is the zero vector in V and 1_L is the supremum element in L

$$5. \lambda^h(0_F) = \lambda^h(1_F) = 1_L$$

The set all multi-fuzzy vector space of V over the multi-fuzzy field λ of F under T-norm T with respect to the function h is denoted by $MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$.

Here, k denotes the dimension of multi-fuzzy set in V and m is the dimension of multi-fuzzy field λ of F , and T denotes the T-norm on L .

We say that $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ is a multi-fuzzy vector space of V over the multi fuzzy field λ of F with respect to the function h and denote it as $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

The function h is called the bridge function of the space

Theorem 4.2. Let V be a vector space over the field F , and let μ is a multi-fuzzy subset of V with

$\mu(0_V) = 1_L$. If $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$, then

$$\mu(px + qy) \geq T(T(\lambda^h(p)\mu(x)), T(\lambda^h(q), \mu(y))) \text{ for all } x, y \in V \text{ and for all } q, p, \in F \dots (A)$$

Conversely, if the crisp function $h: M \rightarrow L$ is such that $\lambda^h(0_F) = \lambda^h(1_F) = 1_L = (1_{L_1}, 1_{L_2}, \dots, 1_{L_k})$ and satisfies the condition (A), then $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$, where $T = (t_1, t_2, \dots, t_k)$ is idempotent and $L = \prod_{j=1}^k L_j$

Proof. Let $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Now, $\mu(px + qy) \geq T(\mu(px), \mu(qy)) \geq T(T(\lambda^h(p), \mu(x)), T(\lambda^h(q), \mu(y)))$ for all $x, y \in V$ and for all $q, p, \in F$.

Conversely, suppose that multi fuzzy subset μ of V satisfies the condition (A).

Let $p = 1_F = 1$, $q = 1_F = 1$ in the condition (A), then we get:

$$\mu(x + y) \geq T(T(\lambda^h(1), \mu(x)), T(\lambda^h(1), \mu(y))) \dots (B).$$

Consider

$$\begin{aligned} T(\lambda^h(1), \mu(x)) &= (t_1(1_{L_1}, \mu_1(x)), t_2(1_{L_2}, \mu_2(x)), \dots, t_k(1_{L_k}, \mu_k(x))) \\ &= (\mu_1(x), \mu_2(x), \dots, \mu_k(x)) \\ &= \mu(x) \end{aligned}$$

Similarly, we can prove that $T(\lambda^h(1), \mu(y)) = \mu(y)$

By substituting the results into (B), we have:

$$\mu(x + y) \geq T(\mu(x), \mu(y)).$$

Also, from (B), we have $\mu(px) = \mu(px + 0_F \cdot x) \geq T(T(\lambda^h(p), \mu(x)), T(\lambda^h(0_F), \mu(x)))$

$$\text{Consider } T(\lambda^h(0_F), \mu(x)) = T(1_L, \mu(x))$$

$$\begin{aligned} &= (t_1(1_{L_1}, \mu_1(x)), t_2(1_{L_2}, \mu_2(x)), \dots, t_k(1_{L_k}, \mu_k(x))) \\ &= (\mu_1(x), \mu_2(x), \dots, \mu_k(x)) \\ &= \mu(x) \end{aligned}$$

Therefore, $\mu(px) \geq T(T(\lambda^h(p), \mu(x)), \mu(x))$

$$\begin{aligned}
&= T(\lambda^h(p), T(\mu(x), \mu(x))) \\
&= T(\lambda^h(p), \mu(x)), \text{ since } T \text{ is idempotent.}
\end{aligned}$$

Finally,

$$\begin{aligned}
\mu(-x) &\geq T(\lambda^h(-1), \mu(x)) \\
&= T(\lambda^h(1), \mu(x)), \text{ since } \lambda^h(-1) = h(\lambda(-1)) = h(\lambda(1)) = \lambda^h(1) \\
&= T(1_L, \mu(x)) \\
&= \mu(x) \text{ for all } x
\end{aligned}$$

Hence, $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Theorem 4.3. Let V be a vector space over the field F . Let $\mu, \tau \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$, and T is idempotent. Then the intersection $\mu \cap \tau \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Proof

Let V be a vector space over the field F and let $x, y \in V$ and $p \in F$.

$$\begin{aligned}
\text{Then, } (\mu \cap \tau)(x + y) &= T(\mu(x + y), \tau(x + y)) \\
&\geq T(T(\mu(x), \mu(y)), T(\tau(x), \tau(y))) \\
&= T(T(\mu(x), \tau(x)), T(\mu(y), \tau(y))), \text{ by the commutativity and associativity of the operator } T, \text{ which simplifies to} \\
&= T((\mu \cap \tau)(x), (\mu \cap \tau)(y))
\end{aligned}$$

$$\text{Therefore, } (\mu \cap \tau)(x + y) \geq T((\mu \cap \tau)(x), (\mu \cap \tau)(y))$$

$$\begin{aligned}
\text{Now, } (\mu \cap \tau)(px) &= T(\mu(px), \tau(px)) \\
&\geq T(T(\lambda^h(p), \mu(x)), T(\lambda^h(p), \tau(x))) \\
&= T(T(\lambda^h(p), \lambda^h(p)), T(\mu(x), \tau(x))), \text{ by the commutativity and associativity of the operator } T, \text{ which simplifies to} \\
&= T(\lambda^h(p), (\mu \cap \tau)(x))
\end{aligned}$$

$$\text{Also we have } (\mu \cap \tau)(0_V) = T(\mu(0_V), \tau(0_V)) = T(1_L, 1_L) = 1_L$$

Therefore, $\mu \cap \tau \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Theorem 4.4

Let V be a vector space over the field F . Let $\{\mu_j \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T) : j \in J\}$ be an indexed family of multi-fuzzy vector spaces of V over the multi-fuzzy field λ of F and the T -norm is the minimum operator in L . Then their intersection:

$$\mu = \bigcap_{j \in J} \mu_j \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$$

Proof

Let V be a vector space over the field F . let $x, y \in V$ and $p \in F$. Let $\bigwedge_{j \in J}$ denotes the infimum taken over all $j \in J$

$$(\bigcap_{j \in J} \mu_j)(x + y) = \bigwedge_{j \in J} \{\mu_j(x + y) : j \in J\}$$

$$\geq \bigwedge_{j \in J} T\{\mu_j(x), \mu_j(y)\}$$

$$= T\{\bigwedge_{j \in J} \mu_j(x), \bigwedge_{j \in J} \mu_j(y)\}$$

$$= T\left\{\left(\bigcap_{j \in J} \mu_j\right)(x), \left(\bigcap_{j \in J} \mu_j\right)(y)\right\}$$

$$(\bigcap_{j \in J} \mu_j)(py) = \bigwedge_{j \in J} \{\mu_j(py) : j \in J\}$$

$$\geq \bigwedge_{j \in J} \{\mu_j(py)\}$$

$$\geq \bigwedge_{j \in J} T\{\lambda^h(p), \mu_j(x)\}$$

$$= T \left\{ \lambda^h(p), \left(\bigwedge_{j \in J} \mu_j(x) \right) \right\}$$

$$= T \left\{ \lambda^h(p), \left(\bigcap_{j \in J} \mu_j \right)(x) \right\}$$

There fore $\mu = \bigcap_{j \in J} \mu_j \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Theorem 4.5

Let V be a vector space over field F and W be a subset of V and let $\mu: V \rightarrow L = \prod_{j=1}^k L_j$ be defined by $\mu(x) = \begin{cases} 1_L, & \text{if } x \in W \\ 0_L, & \text{if } x \notin W \end{cases}$.

Let crisp function h be such that $\lambda^h(p) = 1$ for all p and let T be idempotent. Then, $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$ if and only if W is a subspace of V .

Proof

Let W be a subset of V and $x, y \in W$, $p \in F$, and $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Then, $\mu(x) = 1_L$, $\mu(y) = 1_L$ so that

$$\mu(px + y) \geq T \left(T(\lambda^h(p), \mu(x)), \mu(y) \right) = T(T(\lambda^h(p), 1_L), 1_L) = T(\lambda^h(p), 1_L) = h(\lambda(p))$$

Thus, $px + y \in W$ if $\lambda^h(p) = 1$ for all p .

Therefore, W is a subspace of V if $\lambda^h(p) = 1$ for all p .

Conversely suppose that W is a subspace of V . Consider the multi-fuzzy set

$$\mu(x) = \begin{cases} 1_L, & \text{if } x \in W \\ 0_L, & \text{if } x \notin W \end{cases}$$

For $x, y \in W$ and $p \in F$, we have $px + y \in W$

$$\text{Then, } \mu(px + y) = 1_L = T(1_L, 1_L) = T(T(\lambda^h(p), 1_L), 1_L) = T(\lambda^h(p), \mu(x)), \mu(y))$$

Thus, $\mu \in MFV^{(k,m)}(V, (\lambda, h \in L^M), T)$

Conclusion

This paper analysed the structure and characteristics of a multi-fuzzy vector space $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ of V over the multi-fuzzy field $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of F , when the dimensions of μ and λ are equal. After that we discussed a new type of multi-fuzzy vector space μ over the multi-fuzzy field λ using a bridge function when the dimensions of the multi-fuzzy sets μ and λ differ. Further research is required to investigate the role of different bridge functions in multi-fuzzy vector spaces. This study provides a foundation for future exploration in multi-fuzzy linear algebra and its potential applications.

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