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## Irreducible cyclic codes of length $16p^n$ and their parameters

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### Abstract

This paper derives the expressions for primitive idempotents for semi simple ring  $R_{16p^n} = \frac{GF(q)[x]}{\langle x^{16p^n} - 1 \rangle}$ , where,  $p$  and  $q$  (of type  $16k + 1$ ) are distinct odd primes,  $n$  is an integer that is positive and  $\text{od}(q)$  modulo  $p^n$  is  $\frac{\phi(p^n)}{d}$ . Among other results, we derive the minimum distance bounds and the generating polynomials for the related irreducible cyclic codes.

**Keywords:** Irreducible cyclic codes, cyclotomic cosets, generating and minimal polynomials, minimum distance bounds

### Introduction

The group algebra  $FC_{16p^n}$ , where  $F$  is a  $q$ -order field and  $C_{16p^n}$  is cyclic group of order  $16p^n$  with  $\gcd(q, 16p) = 1$  is semi-simple and collection of primitive idempotents and  $q$ -cyclotomic cosets modulo  $16p^n$  has equal cardinality. The minimal cyclic codes of length  $m$  for  $m = 2, 4, p^n, 2p^n$  were found in [6, 8]. This is the case when the  $\text{Od}(q)$  modulo  $m$  is  $\phi(m)$ . In [7], minimal codes for quadratic residue of length  $p^n$  were computed. In [1, 2] minimal cyclic codes of length  $p^n q$ , were found, where the odd primes  $p$  and  $q$  are distinct. Primitive idempotents in  $FC_{8p^n}$  where  $q$  the type of  $q$  is  $8k + 1$  and  $\text{Od}(q)$  modulo  $8p^n$  is  $\frac{\phi(p^n)}{2}$  were obtained in (4) and when  $q$  is primitive root modulo  $p^n$  are obtained in [5]. Primitive idempotents in a semisimple ring were obtained in [3]. Generators idempotents in Semi Simple Ring  $FC_{16p^n}$ , where,  $q$  is of the type  $16k + 1$  and  $\text{Od}(q)$  modulo  $16p^n$  is  $\frac{\phi(p^n)}{2}$  were obtained in [9]. In this paper, we generalize the result of [4, 5] by taking order of  $q$  modulo  $p^n$  as  $\frac{\phi(p^n)}{d}$ ,  $q \equiv 1 \pmod{16}$ ,  $m = 16p^n$  and obtained the irreducible cyclic codes of length  $16p^n$  over  $GF(q)$ . Here, section 2 contains the  $q$ -cyclotomic cosets modulo  $16p^n$  while section 3 has the corresponding primitive idempotents. We acquired the dimensions and generating polynomials in section 4 and the bounds for minimum distance of the codes in section 5. The different parameters for these codes are covered with an example at the end.

**Preliminaries:** Throughout this study, in some extension field of  $F$  we consider  $\alpha$  to be  $16p^n$ th root of unity. For  $\alpha^s$ ,  $s \in \Omega_s$ ,  $m_s(x)$  is the minimal polynomial and the minimal ideal  $M_s$  in  $R_{16p^n}$ , is generated by  $\frac{x^{16p^n} - 1}{m_s(x)}$ . The primitive idempotent  $P_s(x)$  in  $R_{16p^n}$  cor

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$P_s(x)$  in  $R_{16p^n}$  corresponding to the cyclotomic coset  $C_s$ , given by  $P_s(x) = \frac{1}{16p^n} \sum_{t=0}^{16p^n-1} \varepsilon_i^s x^s$  where  $\varepsilon_i^s = \sum_{s \in \Omega_s} \alpha^{-is}$  and  $\bar{C}_s = \sum_{s \in \Omega_s} x^s$ .

## 2. Cyclotomic Cosets

**Lemma 2.1.** If  $\frac{\phi(p^n)}{d}$  is multiplicative  $\text{Od}(q)$  modulo  $p^n$ , then  $\text{Od}(q)$  modulo  $p^{n-i}$  is  $\frac{\phi(p^{n-i})}{d}$  for all  $i$ ,  $0 \leq i \leq n-1$ .

**Lemma 2.2.** If  $\frac{\phi(p^n)}{d}$  is the  $\text{Od}(q)$  modulo  $p^n$ , then for  $0 \leq i \leq n-1$ ,  $\text{Od}(q)$  modulo  $2^j p^{n-i}$  is  $\frac{\phi(p^{n-i})}{d}$  for all  $j$ ,  $1 \leq j \leq 4$ .

**Lemma 2.3.** If  $\text{Od}(q)$  modulo  $16p^{n-i}$  is  $\frac{\phi(p^{n-i})}{d}$ , then so is of  $q$  modulo  $p^{n-i}$ ; for all  $i$ ,  $1 \leq i \leq n-1$ .

**Lemma 2.4.** For all  $i$ ,  $0 \leq i \leq n-1$  and  $0 \leq k \leq \frac{\phi(p^{n-i})}{d} - 1$ ,  $T \not\equiv q^k \pmod{16p^{n-i}}$  for  $T = \lambda, \mu, \nu, \chi, \psi, \tau, \pi$  where  $\lambda = 1 + 2p^n$ ,  $\mu = 1 + 4p^n$ ,  $\nu = 1 + 6p^n$ ,  $\chi = 1 + 8p^n$ ,  $\psi = 1 + 10p^n$ ,  $\tau = 1 + 12p^n$ ,  $\pi = 1 + 14p^n$ .

**Lemma 2.5.** For any odd prime  $p$  of the form  $4k+1$ , there is some primitive root  $g$ ,  $1 < g < 16p$  with order 2, 2, 4 and modulo 4, 8, 16 respectively and the same for  $p$  of the form  $4k+3$  is 2, 2, 2. Further,  $g \notin \{1, q, q^2, \dots, q^{\frac{\phi(p)-1}{d}}\}$ .

**Lemma 2.6.** A fixed integer  $g$  exists that satisfies  $\gcd(g, 2pq) = 1$ ,  $1 < g < 16p$ , where,  $k \in \{0, 1, \dots, \frac{\phi(p)}{d} - 1\}$  such that the system  $\bigcup_{k=0}^{d-1} \{g^k, g^k q, g^k q^2, \dots, g^k q^{\frac{\phi(p)-1}{d}-1}\}$  is a RRS (Reduced Residue System) modulo  $p^{n-j}$  and the system  $\bigcup_{k=0}^{d-1} \bigcup_{t \in A_1} \{tg^k, tg^k q, \dots, tg^k q^{\frac{\phi(p)-1}{d}-1}\}$  where  $A_1 = \{1, \lambda, \mu, \nu, \psi, \tau, \pi\}$  form a RRS modulo  $16p^{n-j}$ .

**Proof.** Since  $\text{od}(q)$  modulo  $p$  is  $\frac{\phi(p)}{d}$  and the set  $\{q^i : 0 \leq i \leq \frac{\phi(p)}{d} - 1\}$  is incongruent modulo  $p$ , so there is some  $g$  which is co-prime with  $2pq$ ,  $1 < g < 16p$ ,  $g \not\equiv q^k \pmod{p}$  for where  $k \in \{0, 1, \dots, \frac{\phi(p)}{d} - 1\}$ . Then the system

$\bigcup_{k=0}^{d-1} \{g^k, g^k q, g^k q^2, \dots, g^k q^{\frac{\phi(p^{n-j})-1}{d}-1}\}$  is a RRS modulo  $p$ . Now, for  $0 \leq k \leq \frac{\phi(p)}{d} - 1$ ,  $g \not\equiv q^k \pmod{p}$ , implies

$g \not\equiv q^k \pmod{p^{n-j}}$ , for  $0 \leq k \leq \frac{\phi(p^{n-j})}{d} - 1$ . Hence the system  $\bigcup_{k=0}^{d-1} \{g^k, g^k q, g^k q^2, \dots, g^k q^{\frac{\phi(p^{n-j})-1}{d}-1}\}$  is a RRS modulo

$p^{n-j}$ ,  $0 \leq j \leq n-1$ . Due to similar reasoning the set  $\bigcup_{k=0}^{d-1} \bigcup_{t \in A_1} \{tg^k, tg^k q, \dots, tg^k q^{\frac{\phi(p)-1}{d}-1}\}$  is a RRS modulo  $16p^{n-j}$ .

**Theorem 2.1.** For  $q \equiv 1 \pmod{16}$  there are  $16(nd+1)$   $q$ -cyclotomic cosets modulo  $16p^n$  given by  $\Omega_{rp^n} = \{rp^n\}$ ,

$\Omega_{sg^k p^i} = \{sg^k p^i, sg^k p^i q, fg^k p^i q^2, \dots, sg^k p^i q^{\frac{\phi(p)-1}{d}-1}\}$  for  $0 \leq i \leq n-1$ ,  $0 \leq k \leq d-1$ ;  $0 \leq r \leq 15$ ;  $s = 1, 2, 4, 8, 16$ ,  $\lambda, 2\lambda, \mu, 2\mu, \nu, 2\nu, \chi, \psi, \tau, \pi$  and  $g$  is given by Lemma 2.5.

**Proof.**  $\Omega_0 = \{0\}$  is trivial. Since,  $q \equiv 1 \pmod{16}$ . Therefore,  $p^n q \equiv p^n \pmod{16p^n}$  and hence  $\Omega_{p^n} = \{p^n\}$ . Similarly,  $\Omega_{3p^n} = \{3p^n\}$ ,  $\Omega_{5p^n} = \{5p^n\}$ ,  $\Omega_{7p^n} = \{7p^n\}$ ,  $\Omega_{9p^n} = \{9p^n\}$ ,  $\Omega_{11p^n} = \{11p^n\}$ ,  $\Omega_{13p^n} = \{13p^n\}$ ,  $\Omega_{15p^n} = \{15p^n\}$ . Also,  $q \equiv 1 \pmod{8}$

$$\Rightarrow 2p^n q \equiv 2p^n \pmod{16p^n} \Rightarrow \Omega_{2p^n} = \{2p^n\}. \text{ Similarly, } \Omega_{4p^n} = \{4p^n\}, \quad \Omega_{6p^n} = \{6p^n\}, \quad \Omega_{8p^n} = \{8p^n\}, \\ \Omega_{10p^n} = \{10p^n\}, \Omega_{12p^n} = \{12p^n\}, \Omega_{14p^n} = \{14p^n\}.$$

$$\text{By Lemma 2.2, } q^{\frac{\phi(p^{n-i})-1}{d}} \equiv 1 \pmod{16p^{n-i}} \Rightarrow p^i q^{\frac{\phi(p^{n-i})-1}{d}} \equiv p^i \pmod{16p^n}$$

$$\Rightarrow \Omega_{p^i} = \{p^i, p^i q, \dots, p^i q^{\frac{\phi(p^{n-i})-1}{d}}\}$$

Also,  $g^j \equiv g^k \pmod{p^{n-j}}$  for  $0 \leq k \leq \frac{\phi(p^{n-i})}{d} - 1$ ,  $\Omega_{g^j p^k}$  and  $\Omega_{g^l p^m}$  are pairwise disjoint whenever  $j \neq l$  or  $k \neq m$ .

For fixed  $k$  and  $i$ ,  $0 \leq j \leq n-1$ ,  $g^k p^i q^{\frac{\phi(p^{n-i})}{d}} \equiv g^j p^i \pmod{p^{n-j}}$ . Thus, the cyclotomic cosets containing  $g^k p^i$  is  $\Omega_{g^k p^i} = \{g^k p^i, g^k p^i q, \dots, g^k p^i q^{\frac{\phi(p^{n-i})}{d}-1}\}$ . Since  $\text{od}(q)$  modulo  $2p^{n-i}$ ,  $4p^{n-i}$ ,  $8p^{n-i}$  and  $16p^{n-i}$  is also  $\frac{\phi(p^{n-i})}{d}$ , so,

$$q^{\frac{\phi(p)}{d}} \equiv 1 \pmod{16p^{n-i}}$$

$$\Rightarrow sp^i q^{\frac{\phi(p)}{d}} \equiv sp^i \pmod{16p^n}$$

Since,  $g^j \not\equiv g^k \pmod{p^{n-j}}$  so  $\Omega_{sg^j p^k}$  and  $\Omega_{sg^l p^m}$  are respectively pairwise disjoint whenever  $k \neq l$  or  $j \neq m$ . For fixed  $k$

and  $i$ ,  $sg^k p^i q^{\frac{\phi(p^{n-i})}{d}} \equiv sg^j p^i \pmod{p^{n-j}}$ . Thus, the cyclotomic coset containing  $sg^k p^i$  are

$$\Omega_{sg^k p^i} = \{sg^k p^i, sg^k p^i q, sg^k p^i q^2, \dots, sg^k p^i q^{\frac{\phi(p^{n-i})}{d}-1}\} \text{ respectively. Here, } |\Omega_{rp^n}| = 1 \text{ and } |\Omega_{sg^k p^i}| = \frac{\phi(p^{n-i})}{d}.$$

$$\text{Hence, } \sum_{r=0}^{15} |\Omega_{rp^n}| + \sum_{k=0}^{d-1} \sum_{i=0}^{n-1} |\Omega_{sg^k p^i}|.$$

$$\sum_{k=0}^{d-1} \sum_{i=0}^{n-1} |\Omega_{sg^k p^i}| = \sum_{i=0}^{n-1} \phi(p^{n-i}) = \phi(p^n) + \phi(p^{n-1}) + \dots + \phi(p)$$

$$= p^n - p^{n-1} + p^{n-1} - p^{n-2} + \dots - 1 = p^n - 1$$

$$\text{Hence, } \sum_{r=0}^{15} |\Omega_{rp^n}| + \sum_{k=0}^{d-1} \sum_{i=0}^{n-1} \left[ \sum_s |\Omega_{sg^k p^i}| \right] = 16 + 16(p^n - 1) = 16p^n.$$

### 3. Primitive Idempotents

#### 3.1 Primitive Idempotents corresponding to $\Omega_{rp^n}$ where $0 \leq r \leq 15$

**Lemma 3.1.1.** For  $q \equiv 1 \pmod{16}$ ,  $\Omega_{rp^n} = \Omega_{(16-r)p^n}$  where  $0 \leq r \leq 15$ .

**Theorem 3.1.1.** For  $q \equiv 1 \pmod{16}$  the explicit expression for the primitive idempotents  $P_{rp^n}$   $0 \leq r \leq 15$  in  $R_{16p^n}$  are given by

$$P_a(x) = \frac{1}{16p^n} \left[ \sum_{r=0}^{15} \alpha^{(16-t)rp^{2n}} \bar{C}_{rp^n} + \sum_{k=0}^{d-1} \sum_{i=0}^{n-1} \left[ \sum_{a \in A} \alpha^{(16-t)ag^k p^{n+i}} \bar{C}_{ag^k p^i} \right] \right]$$

**Proof.** By definition,

$$P_i(x) = \frac{1}{16p^n} \left[ \sum_{r=0}^{15} k_{rp^n}^t \bar{C}_{rp^n} + \sum_{k=0}^{d-1} \sum_{i=0}^{n-1} \sum_s \left\{ k_{sg^k p^i} \bar{C}_{sg^k p^i} \right\} \right] \dots (2)$$

(i) Expression for  $P_0(x)$  :-

$$K_k^t = \sum_{s \in \Omega_t} \alpha^{-ks}. \text{ Take } s = 0, \text{ to get } K^{k0} = \sum_{s \in \Omega_0} \alpha^0 = 1, \forall 0 \leq k \leq 16p^n - 1.$$

$$\text{Therefore, } P_o(x) = \frac{1}{16p^n} \left[ \sum_{r=0}^{15} \bar{C}_{rp^n} + \sum_{k=0}^{d-1} \sum_{i=0}^{n-1} \left[ \sum_s \bar{C}_{sg^k p^i} \right] \right]$$

(ii) Evaluation of  $P_{p^n}(x)$  :-

$$\text{Since, } \Omega_{p^n} = -\Omega_{15p^n}, K_k^{p^n} = \sum_{s \in \Omega_{p^n}} \alpha^{-ks} = \alpha^{-p^n k} = \alpha^{15p^n k}$$

$$K_0^{p^n} = -K_{8p^n}^{p^n} = 1, K_{p^n}^{p^n} = -K_{9p^n}^{p^n} = -\alpha^{7p^{2n}}, K_{2p^n}^{p^n} = -K_{10p^n}^{p^n} = -\alpha^{6p^{2n}}, K_{3p^n}^{p^n} = -K_{11p^n}^{p^n} = -\alpha^{5p^{2n}},$$

$$K_{4p^n}^{p^n} = -K_{12p^n}^{p^n} = -\alpha^{4p^{2n}}, K_{5p^n}^{p^n} = -K_{13p^n}^{p^n} = -\alpha^{3p^{2n}}, K_{6p^n}^{p^n} = -K_{14p^n}^{p^n} = -\alpha^{2p^{2n}}, K_{7p^n}^{p^n} = -K_{15p^n}^{p^n} = -\alpha^{p^{2n}},$$

$$K_{g^k p^i}^{p^n} = -K_{g^k p^i}^{p^n} = -\alpha^{7g^k p^{n+i}}, K_{2g^k p^i}^{p^n} = -K_{2\mu g^k p^i}^{p^n} = -\alpha^{6g^k p^{n+i}},$$

$$K_{4g^k p^i}^{p^n} = -K_{4\lambda g^k p^i}^{p^n} = -\alpha^{4g^k p^{n+i}}, K_{8g^k p^i}^{p^n} = -K_{16g^k p^i}^{p^n} = -1, K_{\lambda g^k p^i}^{p^n} = -\alpha^{7\lambda g^k p^{n+i}}, K_{2\lambda g^k p^i}^{p^n} = -\alpha^{6\lambda g^k p^{n+i}}$$

$$K_{\tau g^k p^i}^{p^n} = -\alpha^{7\tau g^k p^{n+i}}, K_{\mu g^k p^i}^{p^n} = -\alpha^{7\mu g^k p^{n+i}}, K_{\nu g^k p^i}^{p^n} = -\alpha^{7\nu g^k p^{n+i}}, K_{2\nu g^k p^i}^{p^n} = -\alpha^{6\nu g^k p^{n+i}}, K_{\psi g^k p^i}^{p^n} = -\alpha^{7\psi g^k p^{n+i}},$$

$$K_{\pi g^k p^i}^{p^n} = -\alpha^{7\pi g^k p^{n+i}}$$

Using these in (2) the expression for  $P_{p^n}(x)$  can be obtained.

Similarly, all the expressions for  $P_{rp^n}(x)$ ,  $2 \leq r \leq 15$  can be obtained.

### 3.2 Primitive Idempotents corresponding to $\Omega_{tg^k p^i}$ , where,

$$t = \{1, 2, 4, 8, 16, \lambda, \mu, \nu, \chi, \psi, \tau, \lambda, 2\lambda, 2\mu, 2\nu, 4\lambda\}$$

**Notation:** For  $0 \leq j \leq n-1$  and  $0 \leq k \leq d-1$ , define  $A_j^{(k)} = p^j \sum_{s \in \Omega_{16g^k p^j}} \alpha^s$ ,  $B_j^{(k)} = p^j \sum_{s \in \Omega_{8g^k p^j}} \alpha^s$ ,  $C_j^{(k)} = p^j \sum_{s \in \Omega_{4g^k p^j}} \alpha^s$ ,  
 $D_j^{(k)} = p^j \sum_{s \in \Omega_{4g^k p^j}} \alpha^s$ ,  $E_j^{(k)} = p^j \sum_{s \in \Omega_{4\lambda g^k p^j}} \alpha^s$ ,  $F_j^{(k)} = p^j \sum_{s \in \Omega_{g^k p^j}} \alpha^s$ ,  $G_j^{(k)} = p^j \sum_{s \in \Omega_{\lambda g^k p^j}} \alpha^s$ ,  $H_j^{(k)} = p^j \sum_{s \in \Omega_{\mu g^k p^j}} \alpha^s$ ,  
 $I_j^{(k)} = p^j \sum_{s \in \Omega_{\nu g^k p^j}} \alpha^s$ .

$$\text{Since, } (A_j^{(k)})^q = (p^j \sum_{s \in \Omega_{16g^k p^j}} \alpha^s)^q = (p^j)^q (\sum_{s \in \Omega_{16g^k p^j}} \alpha^s)^q = (p^j)^q (\sum_{s \in \Omega_{16g^k p^j}} \alpha^{qs}) = (p^j)^q (\sum_{s \in \Omega_{16g^k p^j}} \alpha^s) \text{ and } (p^j)^q = p^j.$$

$$\text{Therefore, } (A_j^{(k)})^q = p^j \sum_{s \in \Omega_{16g^k p^j}} \alpha^s. \text{ Also, } \Omega_{p^i} \text{ is } q\text{-cyclotomic class, so } q\Omega_{p^i} = \Omega_{p^i}. \text{ Hence,}$$

$$(A_j^{(k)})^q = p^j \sum_{s \in \Omega_{16g^k p^j}} \alpha^s = A_j^{(k)}, \text{ so } A_j^{(k)} \in \text{GF}(q). \text{ Similarly, } B_j^{(k)} \in \text{GF}(q), C_j^{(k)} \in \text{GF}(q), D_j^{(k)} \in \text{GF}(q),$$

$$E_j^{(k)} \in \text{GF}(q), F_j^{(k)} \in \text{GF}(q), G_j^{(k)} \in \text{GF}(q), H_j^{(k)} \in \text{GF}(q), I_j^{(k)} \in \text{GF}(q).$$

**Lemma 3.2.1.** For  $q \equiv 1(\text{mod } 16)$  the following hold

(i) If  $\frac{\phi(p^n)}{d}$  is even, then  $\Omega_\pi = -\Omega_1$  for  $p \equiv 1(\text{mod } 4)$  or  $p \equiv 3(\text{mod } 4)$  with  $n$  is even and  $\Omega_\nu = -\Omega_1$  for  $p \equiv 1(\text{mod } 4)$  with  $n$  is odd.

(ii) If  $\frac{\phi(p^n)}{d}$  is odd, then  $\Omega_{\pi g^{d/2}} = -\Omega_1$  for  $p \equiv 1(\text{mod } 4)$  or  $p \equiv 3(\text{mod } 4)$  with  $n$  is odd and  $\Omega_{\pi g^{d/2}} = -\Omega_1$  for  $p \equiv 1(\text{mod } 4)$  with  $n$  is even.

**Lemma 3.2.2.**  $16\lambda\Omega_{p^j} = \Omega_{16\lambda p^j} = \Omega_{16p^j} = -\Omega_{16p^j}.$

**Lemma 3.2.3.**  $\sum_{k=0}^{d-1} A_j^{(k)} = \begin{cases} -p^{n-1} ; & j = n-1 \\ 0 ; & \text{otherwise} \end{cases}$

**Lemma 3.2.4.** For  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$  and  $0 \leq h, k \leq d-1$ ,

$$\sum_{s \in \Omega_{16g^k p^j}} \alpha_{s_1 g^h p^i s} = \begin{cases} \frac{\phi(p^{n-j})}{d} ; & \text{if } i+j \geq n \\ \frac{1}{p^j} A_{i+j}^{(h+k)} ; & \text{if } i+j \leq n-1 \end{cases}$$

for  $s_1 = 1, 2, 4, 8, 16, \lambda, \mu, \nu, \chi, \psi, \tau, \pi, 2\lambda, 4\lambda, 2\mu, 2\nu$ .

**Theorem 3.2.1.** If  $q \equiv 1(\text{mod } 16)$  and  $\frac{\phi(p^n)}{d}$  is even, then there are nd number of primitive idempotents corresponding to  $\Omega_{16g^k p^j}$  in  $R_{16p^n}$  are given by

$$P_{16g^k p^j(x)} = \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} \bar{C}_{p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{ \bar{C}_{ag^h p^i} \} + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{t=0}^{n-j-1} \sum_{a \in A} \{ A_{i+j}^{(h+k)} \bar{C}_{ag^h p^i} \} \right]$$

where for  $q \equiv 1(\text{mod } 16)$ ,  $0 \leq k \leq d-1$  and  $0 \leq j \leq n-1$ ,

(i)  $A_h^{(k)} = 0$ ,  $\forall$   $h$  and  $k$ .

(ii) If  $\frac{d}{2}$  is even, then  $\sum A_{n-1}^{(k)} = \begin{cases} -\frac{(p^{n-1})(1-\sqrt{p})}{2} ; & \text{if } k \text{ is even} \\ -\frac{(p^{n-1})(1+\sqrt{p})}{2} ; & \text{if } k \text{ is odd} \end{cases}$

Further if  $\frac{d}{2}$  is odd, then  $\sum A_{n-1}^{(k)} = \begin{cases} -\frac{(p^{n-1})(1-\sqrt{-p})}{2} ; & \text{if } k \text{ is even} \\ -\frac{(p^{n-1})(1+\sqrt{-p})}{2} ; & \text{if } k \text{ is odd} \end{cases}$

**Proof.** To evaluate  $P_{16g^k p^j}(x)$  replace  $s$  by  $16g^k p^j$  in (2). For this compute,  $K_m^{16g^k p^j}$  for  $m = rp^n$ , where  $0 \leq r \leq 15$  and  $s = 1, 2, 4, 8, \lambda, 2\lambda, 4\lambda, \mu, 2\mu, \nu, 2\nu, \chi, \psi, \tau, \pi$

Since,  $\Omega_{16p^j} = -\Omega_{16p^j}$  by lemmas 3.1 and 3.2

$$K_m^{16g^k p^j} = \sum_{s \in \Omega_{16g^k p^j}} \alpha^{-sm} = \sum_{s \in \Omega_{16g^k p^j}} \alpha^{sm}. \text{ Therefore, } r, s, K_{rp^n}^{16g^k p^j} = \frac{\phi(p^{n-j})}{d}$$

As according to lemma 3.2.4

$$K_m^{16g^k p^j} = \begin{cases} \frac{\phi(p^{n-j})}{d}; & \text{if } i+j \geq n \\ \frac{1}{p^j} A_{i+j}^{(h+k)}; & \text{if } i+j \leq n-1 \end{cases}$$

Using these in (2) we get the expression for  $P_{16g^k p^j}(x)$ .

**Lemma 3.2.6.** For  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$  and  $0 \leq h, k \leq d-1$ ,

$$\sum_{s \in \Omega_{8g^k p^j}} \alpha^{s_1 g^h p^j s} = \begin{cases} \frac{-\phi(p^{n-j})}{d}; & \text{if } i+j \geq n \\ \frac{1}{p_j} B_{i+j}^{(h+k)}; & \text{if } i+j \leq n-1 \end{cases} \text{ for } s_1 = 1, \lambda, \mu, \nu, \chi, \psi, \tau, \pi.$$

**Lemma 3.2.7.** For  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$  and  $0 \leq h, k \leq d-1$ ,

$$\sum_{s \in \Omega_{4g^k p^j}} \alpha^{g^h p^j s} = \begin{cases} \frac{\phi(p^{n-j})}{d} \alpha^{4g^{h+k} p^{i+j}}; & \text{if } i+j \geq n \\ \frac{1}{p_j} C_{i+j}^{(h+k)}; & \text{if } i+j \leq n-1 \end{cases}$$

**Lemma 3.2.8.** For  $0 \leq i \leq n$ ,  $0 \leq j \leq n-1$  and  $0 \leq h, k \leq d-1$ ,

$$\sum_{s \in \Omega_{g^k p^j}} \alpha^{g^h p^j s} = \begin{cases} \frac{\phi(p^{n-j})}{d} \alpha^{g^{h+k} p^{i+j}}; & \text{if } i+j \geq n \\ \frac{1}{p_j} F_{i+j}^{(h+k)}; & \text{if } i+j \leq n-1 \end{cases}$$

**Theorem 3.2.2.** If  $q \equiv 1 \pmod{16}$  and  $\frac{\phi(p^n)}{d}$  is even, then there are nd number of primitive idempotents corresponding to

$$P_{8g^k p^j}(x) = \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15-0} (-1)^t \bar{C}_{\varphi^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{ (-1)^a \bar{C}_{ag^h p^i} \} + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-1-j} \{ A_{i+j}^{(h+k)} \sum_{a \in A_1} \bar{C}_{ag^h p^i} + B_{i+j}^{(h+k)} \sum_{a \in A_2} \bar{C}_{ag^h p^i} \} \right];$$

$A_1 = \{2, 4, 8, 16, 2\lambda, 4\lambda, 2\mu, 2\nu\}$ ,  $A_2 = \{1, \lambda, \mu, \nu, \chi, \psi, \tau, \pi\}$ , where for  $0 \leq k \leq d-1$  and  $0 \leq j \leq n-1$

(i)  $B_j^{(k)} = 0$ .

$$(ii) \text{ If } \frac{d}{2} \text{ is even, then } \sum B_{n-1}^{(k)} = \begin{cases} \frac{(p^{n-1})(1+\sqrt{p})}{2}; & \text{if } k \text{ is even} \\ \frac{(p^{n-1})(1-\sqrt{p})}{2}; & \text{if } k \text{ is odd} \end{cases}$$

$$\text{Further if } \frac{d}{2} \text{ is odd, then } \sum B_{n-1}^{(k)} = \begin{cases} \frac{(p^{n-1})(1+\sqrt{-p})}{2}; & \text{if } k \text{ is even} \\ \frac{(p^{n-1})(1-\sqrt{-p})}{2}; & \text{if } k \text{ is odd} \end{cases}$$

**Theorem 3.2.3.** If  $\frac{\phi(p^n)}{d}$  is even, then there are 4nd number of primitive idempotents in  $R_{16p^n}$  are given by

$$\begin{aligned} P_{4g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15-0} (-1)^t \alpha^{4tg^k p^{n+j}} \bar{C}_{p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{(-1)^a \alpha^{4ag^{h+k} p^{n+j}} \bar{C}_{ag^h p^i}\} \right. \\ &\quad \left. + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-1-j} \{A_{i+j}^{(h+k)} \sum_{b \in B_1} \bar{C}_{bg^h p^i} + B_{i+j}^{(h+k)} \sum_{b \in B_2} \bar{C}_{bg^h p^i} - C_{i+j}^{(h+k)} \sum_{b \in B_4} \bar{C}_{bg^h p^i}\} \right]; \\ P_{4\lambda g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15-0} (-1)^t \alpha^{4tg^k p^{n+j}} \bar{C}_{p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{\alpha^{4ag^{h+k} p^{n+j}} \bar{C}_{ag^h p^i}\} \right. \\ &\quad \left. + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-1-j} \{A_{i+j}^{(h+k)} \sum_{b \in B_1} \bar{C}_{bg^h p^i} + B_{i+j}^{(h+k)} \sum_{b \in B_2} \bar{C}_{bg^h p^i} + C_{i+j}^{(h+k)} \sum_{b \in B_3} \bar{C}_{bg^h p^i} - C_{i+j}^{(h+k)} \sum_{b \in B_4} \bar{C}_{bg^h p^i}\} \right]; \\ B_1 &= \{4, 8, 16, 4\lambda\}, B_2 = \{2, 2\lambda, 2\mu, 2\nu\}, B_3 = \{1, \mu, \chi, \tau\}, B_4 = \{\lambda, \nu, \psi, \pi\} \end{aligned}$$

where for  $q \equiv 1(\text{mod } 8)$  and  $\frac{\phi(p^n)}{d}$  is even then

(i) For  $0 \leq k \leq d-1$  and  $0 \leq j \leq n-1$ ,

$$(ii) \text{ If } \frac{d}{2} \text{ is even, then } \sum C_{n-1}^{(k)} = \begin{cases} \frac{\sqrt{-2p^{2n-2} - p^{2n-1}} + 2\sqrt{-2p^{2n-1}}}{4}; & \text{if } k \text{ is even} \\ \frac{\sqrt{-2p^{2n-2} - p^{2n-1}} - 2\sqrt{-2p^{2n-1}}}{4}; & \text{if } k \text{ is odd} \end{cases}$$

$$\text{Further if } \frac{d}{2} \text{ is odd, then } \sum C_{n-1}^{(k)} = \begin{cases} \frac{\sqrt{-2p^{2n-2} - p^{2n-1}} + 2\sqrt{2p^{2n-1}}}{4}; & \text{if } k \text{ is even} \\ \frac{\sqrt{-2p^{2n-2} - p^{2n-1}} - 2\sqrt{-2p^{2n-1}}}{4}; & \text{if } k \text{ is odd} \end{cases}$$

**Theorem 3.2.4.** If  $\frac{\phi(p^n)}{d}$  is even, then there are 2nd number of primitive idempotents in  $R_{16p^n}$  are given by

$$\begin{aligned} P_{2g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} (-1)^t \alpha^{2tg^k p^{n+j}} \bar{C}_{p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{(-1)^a \alpha^{2\lambda g^{h+k} p^{n+j}} \bar{C}_{ag^h p^i}\} \right. \\ &\quad \left. + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-1-j} \{-D_{i+j}^{h+k} \bar{C}_{g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4g^h p^i} + A_{i+j}^{h+k} \bar{C}_{8g^h p^i} + A_{i+j}^{h+k} \bar{C}_{16g^h p^i} - E_{i+j}^{h+k} \bar{C}_{\lambda g^h p^i}\} \right] \end{aligned}$$

$$\begin{aligned}
& +D_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} + E_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} - E_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} + E_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} \\
& - C_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} \} \\
P_{2\lambda g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} (-1)^t \alpha^{2tg^k p^{n+j}} \bar{C}_{\tau p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{ (-1)^a \alpha^{2g^{h+k} p^{i+j}} \bar{C}_{ag^h p^i} \} \right. \\
& + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-j-1} \{ -E_{i+j}^{h+k} \bar{C}_{g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4g^h p^i} + A_{i+j}^{h+k} \bar{C}_{8g^h p^i} + A_{i+j}^{h+k} \bar{C}_{16g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\lambda g^h p^i} \\
& + E_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} - E_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} + E_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} \\
& \left. + C_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} \} \right] \\
P_{2\mu g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} \alpha^{2\lambda g^k p^{n+j}} \bar{C}_{\tau p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{ \alpha^{2\lambda g^{h+k} p^{i+j}} \bar{C}_{ag^h p^i} \} \right. \\
& + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-j-1} \{ E_{i+j}^{h+k} \bar{C}_{g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4g^h p^i} + A_{i+j}^{h+k} \bar{C}_{8g^h p^i} + A_{i+j}^{h+k} \bar{C}_{16g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\lambda g^h p^i} \\
& - E_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} + E_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} - E_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} \\
& \left. + C_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} - B_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} \} \right] \\
P_{2\nu g^k p^j}(x) &= \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} \alpha^{2tg^k p^{n+j}} \bar{C}_{\tau p^n} \right\} + \frac{\phi(p^{n-j})}{d} \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \{ \alpha^{2g^{h+k} p^{i+j}} \bar{C}_{ag^h p^i} \} \right. \\
& + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-j-1} \{ E_{i+j}^{h+k} \bar{C}_{g^h p^i} + C_{i+j}^{h+k} \bar{C}_{2g^h p^i} + B_{i+j}^{h+k} \bar{C}_{4g^h p^i} + A_{i+j}^{h+k} \bar{C}_{8g^h p^i} + A_{i+j}^{h+k} \bar{C}_{16g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\lambda g^h p^i} \\
& - E_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} + E_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} + D_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} - E_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} - D_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} \\
& \left. + C_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} - C_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} - B_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} \} \right]
\end{aligned}$$

where for  $q \equiv 1 \pmod{16}$ ,  $0 \leq k \leq d-1$  and  $0 \leq j \leq n-1$ ,

(i)  $D_j^{(k)} = 0$ ,  $E_j^{(k)} = 0$ ,  $\forall j$  and  $k$ .

(ii) If  $\frac{d}{2}$  is even, then

$$\sum D_{n-1}^{(k)} = \begin{cases} \frac{-p^{2n-1} - p^{2(n-1)}}{2}; & \text{if } k \text{ is even} \\ \frac{-p^{2n-1} + p^{2(n-1)}}{2}; & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad \sum E_{n-1}^{(k)} = \begin{cases} -p^{2n-1}; & \text{if } k \text{ is even} \\ p^{2n-1}; & \text{if } k \text{ is odd} \end{cases}$$

(iii) If  $\frac{d}{2}$  is odd, then

$$\sum D_{n-1}^{(k)} = \begin{cases} \frac{p^{2n-1} + p^{2(n-1)}}{2}; & \text{if } k \text{ is even} \\ \frac{p^{2n-1} - p^{2(n-1)}}{2}; & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad \sum E_{n-1}^{(k)} = \begin{cases} -p^{2n-1} - p^{2n-1}; & \text{if } k \text{ is even} \\ p^{2n-1} - p^{2n-1}; & \text{if } k \text{ is odd} \end{cases}$$

**Theorem 3.2.5.** If  $\frac{\phi(p^n)}{d}$  is even, then there are 2nd number of primitive idempotents in  $R_{16p^n}$  are given by

[illegible]

$$\begin{aligned}
& -E_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} + C_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} - I_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} - D_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} + H_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} - E_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} - G_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} \\
& + F_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} + I_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} - H_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} \} ] \\
P_{\pi g^h p^j}(x) = & \frac{1}{16p^n} \left[ \frac{\phi(p^{n-j})}{d} \left\{ \sum_{t=0}^{15} (-1)^t \alpha^{p^{n+j}} \bar{C}_{\eta^t p^n} \right\} + \frac{\phi(p^{n-j})}{d} \left\{ \sum_{h=0}^{d-1} \sum_{i=n-j}^{n-1} \sum_{a \in A} \alpha^{ag^{h+k} p^{i+j}} \bar{C}_{ag^h p^i} \right\} \right. \\
& + \frac{1}{p^j} \sum_{h=0}^{d-1} \sum_{i=0}^{n-j-1} \{ I_{i+j}^{h+k} \bar{C}_{g^h p^i} + E_{i+j}^{h+k} \bar{C}_{2g^h p^i} + C_{i+j}^{h+k} \bar{C}_{4g^h p^i} + B_{i+j}^{h+k} \bar{C}_{8g^h p^i} + A_{i+j}^{h+k} \bar{C}_{16g^h p^i} + G_{i+j}^{h+k} \bar{C}_{\lambda g^h p^i} \\
& + D_{i+j}^{h+k} \bar{C}_{2\lambda g^h p^i} - C_{i+j}^{h+k} \bar{C}_{4\lambda g^h p^i} + H_{i+j}^{h+k} \bar{C}_{\mu g^h p^i} - E_{i+j}^{h+k} \bar{C}_{2\mu g^h p^i} + I_{i+j}^{h+k} \bar{C}_{\nu g^h p^i} - D_{i+j}^{h+k} \bar{C}_{2\nu g^h p^i} - F_{i+j}^{h+k} \bar{C}_{\chi g^h p^i} \\
& \left. - G_{i+j}^{h+k} \bar{C}_{\psi g^h p^i} - H_{i+j}^{h+k} \bar{C}_{\tau g^h p^i} - I_{i+j}^{h+k} \bar{C}_{\pi g^h p^i} \} \right]
\end{aligned}$$

where for  $q \equiv 1 \pmod{16}$ ,  $0 \leq k \leq d-1$  and  $0 \leq j \leq n-1$ ,

(i)  $F_j^{(k)} = 0$ ,  $G_j^{(k)} = 0$ ,  $H_j^{(k)} = 0$ ,  $I_j^{(k)} = 0$ ,  $\forall j$  and  $k$ .

(ii) If  $\frac{d}{2}$  is even, then

$$\begin{aligned}
\sum F_{n-1}^{(k)} &= \begin{cases} \frac{-p^{2n-1} + p^{2(n-1)}}{2}; & \text{if } k \text{ is even} \\ \frac{-p^{2n-1} - p^{2(n-1)}}{2}; & \text{if } k \text{ is odd} \end{cases}, \quad \sum G_{n-1}^{(k)} = \begin{cases} -(p^{2n-1}); & \text{if } k \text{ is even} \\ (p^{2n-1}); & \text{if } k \text{ is odd} \end{cases} \\
\sum H_{n-1}^{(k)} &= \begin{cases} -(p^{n-1})(1+p); & \text{if } k \text{ is even} \\ (p^{n-1})(1+p); & \text{if } k \text{ is odd} \end{cases}, \quad \sum I_{n-1}^{(k)} = \begin{cases} (p^{n-1})(1-p); & \text{if } k \text{ is even} \\ -(p^{n-1})(1-p); & \text{if } k \text{ is odd} \end{cases}
\end{aligned}$$

#### 4. Dimension and Generating Polynomials:

The minimal polynomial for  $\alpha^s$  is represented by the polynomial  $m_s(s) = \prod_{s \in \Omega_s} (x - \alpha^s)$ , and For the cyclic code  $M_s$  of

length  $16p^n$  that corresponds to the cyclotomic coset  $\Omega_s$ , the generating polynomial is  $\frac{x^{16p^n} - 1}{m_s(x)}$ . The cardinality of the class

$\Omega_s$  is the same as the dimension of  $M_s$ . Thus the dimension of each of the code  $M_{tp^n}$ , where  $0 \leq t \leq 15$  is 1 and

$M_{4p^j}, M_{8p^j}, M_{16p^j}$  is  $\phi(p^{n-j})$ .

**Theorem 4.1.** (i) The generating polynomial for the code  $M_0, M_{p^n}, M_{2p^n}, M_{3p^n}, M_{4p^n}, M_{5p^n}, M_{6p^n}, M_7, M_{8p^n}, M_{9p^n}, M_{10p^n}, M_{11p^n}, M_{12p^n}, M_{13p^n}, M_{14p^n}, M_{15p^n}$  are

$$\begin{aligned}
& (1+x+x^2+\dots+x^{16p^{n-1}}), \\
& (x^8-1)(x^4+\beta^4)(x^2+\beta^2)(x+\beta)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}), \\
& (x+\beta^2)(x^2+\beta^4)(x^4-1)(x^8+1)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}), \\
& (x+\beta^3)(x^2+\beta^6)(x^4-\beta^4)(x^8-1)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}), \\
& (x^2-1)(x+\beta^4)(x^4+1)(x^8+1)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}), \\
& (x^8-1)(x+\beta^5)(x^2-\beta^2)(x^4+\beta^4)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}), \\
& (x^8+1)(x^4-1)(x^2-\beta^4)(x+\beta^6)(1+x^{16}+x^{32}+\dots+x^{16(p^n-1)}),
\end{aligned}$$

$$\begin{aligned}
& (x^8 - 1)(x^4 - \beta^4)(x^2 - \beta^6)(x + \beta^7)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 + 1)(x^4 + 1)(x^2 + 1)(x - 1)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^4 + \beta^4)(x^2 + \beta^2)(x - \beta)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^4 - 1)(x^2 + \beta^4)(x - \beta^2)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^4 - \beta^4)(x^2 + \beta^6)(x - \beta^3)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 + 1)(x^4 + 1)(x^2 - 1)(x - \beta^4)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^4 + \beta^4)(x^2 - \beta^2)(x - \beta^5)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^2 - \beta^4)(x^4 - 1)(x - \beta^6)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}), \\
& (x^8 - 1)(x^4 - \beta^4)(x^2 + \beta^6)(x - \beta^7)(1 + x^{16} + x^{32} + \dots + x^{16(p^{n-1})}),
\end{aligned}$$

respectively, where  $\alpha^{p^n} = \beta$  is 16<sup>th</sup> root of unity.

(ii) The generating polynomial for the codes  $M_{8g^k p^i}$ ,  $M_{16g^k p^i}$  and  $M_{tg^k p^i}$ , where  $t \in A - \{8, 16\}$  are

$$\begin{aligned}
& (x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + x^{32p^{n-i}} + \dots + x^{16(p^{n-i}(p^i-1)-1)}), \\
& (x^{p^{n-i-1}} - 1)(x^{p^{n-i}} + 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(1 + x^{16p^{n-i}} + x^{32p^{n-i}} + \dots + x^{16(p^{n-i}(p^i-1)-1)}), \text{ and} \\
& (x^{2p^{n-i-1}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1)(x^{2p^{n-i-1}} - 1)(1 + x^{16p^{n-i}} + x^{32p^{n-i}} + \dots + x^{16(p^{n-i}(p^i-1)-1)})
\end{aligned}$$

respectively.

**Proof.** (i) The minimal polynomial for  $\alpha^0, \alpha^{p^n}, \alpha^{2p^n}, \alpha^{3p^n}, \alpha^{4p^n}, \alpha^{5p^n}, \alpha^{6p^n}, \alpha^{7p^n}, \alpha^{8p^n}, \alpha^{9p^n}, \alpha^{10p^n}, \alpha^{11p^n}, \alpha^{12p^n}, \alpha^{13p^n}, \alpha^{14p^n}, \alpha^{15p^n}$  are  $(x - 1), (x - \beta), (x - \beta^2), (x - \beta^3), (x - \beta^4), (x - \beta^5), (x - \beta^6), (x - \beta^7), (x + 1), (x + \beta), (x + \beta^2), (x + \beta^3), (x + \beta^4), (x + \beta^5), (x + \beta^6), (x + \beta^7)$ , respectively.

(ii) The product of minimal polynomial satisfied by  $\alpha^{8p^i}$  and  $\alpha^{8gp^i}$  is  $\frac{x^{p^{n-i}} + 1}{x^{p^{n-i-j}} + 1}$ . Therefore, the generating polynomial for

$$\begin{aligned}
& M_{8p^i} * M_{8gp^i} \text{ is} \\
& (x^{p^{n-i-1}} + 1)(x^{p^{n-i}} - 1)(x^{2p^{n-i}} + 1)(x^{4p^{n-i}} + 1)(x^{8p^{n-i}} + 1) \left( (1 + (x^{16p^{n-i}} + x^{32p^{n-i}} + \dots + x^{16(p^{n-i}(p^i-1)-1)})) \right).
\end{aligned}$$

## 5. Minimum Distance Bounds of the Codes $M_{tp^n}$ , where $0 \leq t \leq 15$

Given a cyclic code  $l$  of length  $m$  generated by  $g(x)$  with a minimum distance of  $d$ , we may

deduce that the code  $\bar{l}$  of length  $mk$  formed by  $g(x)(1 + x^m + x^{2m} + \dots + x^{(k-1)m})$  is a repetition code of  $l$  repeated  $k$  times with a minimum distance of  $dk$ .

**Theorem 5.1.** For  $q \equiv 1 \pmod{16}$  the minimum distance for each codes  $M_{tp^n}$ , where  $t \in A'$  are of  $16p^n$ .

**Proof.** As the generating polynomial of the cyclic code  $M_0$  is  $(1 + x + x^2 + \dots + x^{16p^n-1})$  which is a polynomial of length  $16p^n$ , therefore, this have the minimum distance  $16p^n$ .

Now generated by the polynomial of the code  $M_{p^n}$  is  $(x^8 - 1)(x^4 + \beta^4)(x^2 + \beta^2)(x + \beta)$

$(1 + x^{16} + x^{32} + \dots + x^{16(p^n-1)})$ . If we choose a cyclic code have length 16 generated by the polynomial  $(x^8 - 1)(x^4 + \beta^4)(x^2 + \beta^2)(x + \beta)$  then 16 is the minimum distance of this code. As the cyclic code of length  $16p^n$  having

generating polynomial  $(x^8 - 1)(x^4 + \beta^4)(x^2 + \beta^2)(x + \beta)(1 + x^{16} + x^{32} + \dots + x^{16(p^n-1)})$  is repetition of the cyclic code of length 16 having generating polynomial  $(x^8 - 1)(x^4 + \beta^4)(x^2 + \beta^2)(x + \beta)$  repeated  $p^n$  times, therefore  $16p^n$  is the minimum distance.

Similarly, we can obtain the minimum distance of each irreducible cyclic codes  $M_{tp^n}$  where  $t \in A' - \{0, 1\}$  is  $16p^n$ .

## 6. Example

Irreducible Cyclic Codes of length 48.

Take  $p = 3, n = 1, q = 97$  to find  $g$ , let residue system  $S_3 = \{0, 1, 2\}$ ,  $S_2 = \{0, 1\}$  and  $S_6 = \{0, 1, 2, 3, 4, 5\}$  is modulo 3, modulo 2 and modulo 6 respectively. Since  $\gcd(2, 3) = 1$ , so there exist an integer  $v = 2 \in S_3$  such that  $2v - p = 1$ . Clearly  $a = 2$  is primitive root modulo 3 in  $S_3$ .

So  $g \equiv (2av + tp + 4ap)(\text{mod } 48) \Rightarrow g \equiv (8 + 15 + 24)(\text{mod } 48) \Rightarrow g \equiv 47(\text{mod } 48)$ .

Hence,  $g = 47$  is primitive root modulo 48. Therefore, the cyclotomic cosets modulo 48 are  $\Omega_s = \{s\}$ , where  $s$  is an integer  $0 \leq s \leq 47$ . The expression for primitive idempotents is

$$P_a(x) = \frac{1}{48} \left[ \sum_{t=0}^{47} \{(-48)^{at} (\text{mod } 97) C_t\} \right], \text{ where } a = 0 \text{ to } 47.$$

The minimal polynomials for  $\alpha^t, 0 \leq t \leq 47$  are  $(x - r)$ , where  $\alpha^t \equiv r(\text{mod } 97)$ . The generating polynomials  $\bar{g}_x$  corresponding to each of the cyclotomic cosets are

$$\bar{g}_{3a} = \sum_{t=0}^{47} \{4^{ap^n(47-t)} x^t\}(\text{mod } 97), \text{ where } 0 \leq a \leq 15$$

$$\bar{g}_c = \sum_{t=0}^{47} \{4^{cp^n(47-t)} x^t\}(\text{mod } 97), \text{ where } c = \{3t + 1, 3t + 2, 0 \leq a \leq 15\} - \{8, 16, 32, 40\}$$

$$\bar{g}_d = \sum_{t=0}^{47} \{4^{dp^n(47-t)} x^t\}(\text{mod } 97), \text{ where } d = 8, 16, 32, 40.$$

Dimension and minimum distance bounds for the codes  $M_t$ , where  $0 \leq t \leq 47$ .

| Codes                                                      | Dimension | Minimum Distance Bounds |
|------------------------------------------------------------|-----------|-------------------------|
| $M_{3a}$ where $a = 0, 1$                                  | 15        | 48                      |
| $M_d$ where $d = 8, 16, 32, 40$                            | 1         | $32 \leq d \leq 48$     |
| $M_c$ where $c = \{3t + 1, 3t + 2; 0 \leq a \leq 15\} - d$ | 1         | $16 \leq c \leq 48$     |

**Conclusion:** The cyclotomic cosets and the primitive idempotents in Semi Simple Ring  $FC_{16p^n}$ , where,  $q$  is of the type  $16k + 1$  and  $\text{Od}(q)$  modulo  $16_{p^n}$  are obtained. Also the minimal polynomials, generating polynomials and the minimum distance bounds for the codes are obtained.

**Declaration:** All authors declare that they have no conflicts of interest.

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