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On numerical range distortion under metric similarity in Hilbert space operators

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Abstract

This paper presents an analysis of numerical range transformations under the equivalence relation of metric similarity in Hilbert spaces. Operators $T, S \in B(H)$ are metrically similar if $S = UX^{-1}TXU^*$ for some unitary operator U and positive invertible operator X . We prove that metric similarity constitutes an equivalence relation and establish sharp, optimal bounds for numerical range distortions: $W(S) \subseteq \kappa(X) \cdot \text{conv}(W(T))$ with $\kappa(X) = \|X\| \|X^{-1}\|$, demonstrating optimality through explicit constructions. The work confirms spectral invariance preservation of $\sigma(T)$, $\sigma_e(T)$, $\sigma_p(T)$, and $r(T)$ under metric similarity.

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1. Introduction

The study of numerical ranges and their behavior under various equivalence relations represents a fundamental area in operator theory with applications ranging from quantum mechanics to matrix analysis. Building on the seminal work of Toeplitz^[9] and subsequent developments by Halmos^[3] and others, the numerical range has proven to be a powerful geometric tool for understanding operator properties.

Definition 1.1 (Numerical Range). For $T \in B(H)$, the numerical range is defined as

$$W(T) = \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the Hilbert space H .

The Toeplitz-Hausdorff theorem establishes that $W(T)$ is a convex subset of $\mathbb{C}$, and moreover, $\sigma(T) \subseteq W(T)$, where $\sigma(T)$ denotes the spectrum of T .

Definition 1.2 (Essential Numerical Range). The essential numerical range of an operator $A \in B(H)$ is defined as

$$W_e(A) = \{ \lambda \in \mathbb{C} : \text{there exists an orthonormal sequence } \{x_n\} \text{ with } \langle Ax_n, x_n \rangle \rightarrow \lambda \}.$$

Definition 1.3 (Positive Invertible Operator). An operator A on a Hilbert space H is called positive invertible if it satisfies:

1. A is self-adjoint: $A = A^*$;
2. A is positive: $\langle Ax, x \rangle \geq 0$ for all $x \in H$;
3. A is invertible: there exists a bounded operator A^{-1} such that $AA^{-1} = A^{-1}A = I$.

These conditions are equivalent to requiring $A = A^*$ and $\sigma(A) \subset (0, \infty)$, which implies $\langle Ax, x \rangle > 0$ for all nonzero $x \in H$. Moreover, the square root $A^{\frac{1}{2}}$ exists and is also positive invertible.

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Remark 1.4. The self-adjointness condition in Definition 1.3 is crucial. Without it, an operator satisfying only $\langle Ax, x \rangle \geq 0$ need not have real spectrum or admit a square root. For instance, the operator $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ on $\mathbb{C}^2$ satisfies $\langle Tx, x \rangle = |x_1|^2 + x_1 \bar{x}_2 + |x_2|^2$, which can be made positive for appropriate choices, yet $T \neq T^*$ and does not have the structure of a positive operator.

Example 1.5 (Diagonal Positive Operator). Let $H = \ell^2(\mathbb{N})$, the space of square-summable sequences. Define A by

$$A(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots),$$

where $\lambda_i > 0$ for all $i \in \mathbb{N}$, with $\sup_i \lambda_i < \infty$ and $\inf_i \lambda_i > 0$. Then:

1. A is self-adjoint since $A^* = A$ (diagonal with real entries);
 2. A is positive: $\langle Ax, x \rangle = \sum_i \lambda_i |x_i|^2 \geq 0$;
 3. A is invertible with $A^{-1}(x_1, x_2, \dots) = (\lambda_1^{-1} x_1, \lambda_2^{-1} x_2, \dots)$.
- Thus, A is a positive invertible operator. Note that

$$\|A\| = \sup_i \lambda_i \text{ and } \|A^{-1}\| = \left(\inf_i \lambda_i\right)^{-1}$$

Various equivalence relations on operators have been extensively studied in the literature, including similarity^[1], unitary equivalence^[10], and spectral equivalence^[6]. We now introduce the central notion of this paper.

Definition 1.6 (Metric Similarity). Operators $T, S \in B(H)$ are metrically similar, denoted $S \text{ m.s.} \sim T$, if there exists a factorization

$$S = UX^{-1}TXU^*$$

where U is unitary and X is positive invertible. Equivalently, S is unitarily equivalent to an operator similar to T . The condition number of a positive invertible operator X is defined as

$$\kappa(X) = \|X\| \|X^{-1}\|.$$

This quantity measures the distortion introduced by the transformation $T \mapsto X^{-1}TX$.

2. Preliminary Results

We begin by establishing that metric similarity is indeed an equivalence relation.

Theorem 2.1. Metric similarity is an equivalence relation on $B(H)$.

Proof. We verify the three defining properties of an equivalence relation.

Reflexivity: For any $T \in B(H)$, take $U = I$ and $X = I$. Then

$$T = I \cdot I^{-1} \cdot T \cdot I \cdot I^* = T,$$

so $T \text{ m.s.} \sim T$.

Symmetry: Assume $S \text{ m.s.} \sim T$, so there exist unitary U and positive invertible X with

$$S = UX^{-1}TXU^*.$$

Rearranging

$$U^*SU = X^{-1}TX \implies T = XU^*SUX^{-1}.$$

Let $V = U^*$ (unitary) and $Y = X^{-1}$ (positive invertible since X is). Then

$$T = VY^{-1}SYV^*,$$

proving $T \text{ m.s.} \sim S$.

Transitivity: Assume $S \text{ m.s.} \sim T$ and $T \text{ m.s.} \sim R$. Then there exist unitaries U_1, U_2 and positive invertibles X_1, X_2 such that $S = U_1X_1^{-1}TX_1U_1^*$ and $T = U_2X_2^{-1}RX_2U_2^*$.

Substituting the second into the first:

$$S = U_1X_1^{-1}(U_2X_2^{-1}RX_2U_2^*)X_1U_1^* = (U_1X_1^{-1}U_2X_2^{-1})R(X_2U_2^*X_1U_1^*).$$

Let $A = U_1X_1^{-1}U_2X_2^{-1}$. Then $A^{-1} = X_2U_2^*X_1U_1^*$, and $S = ARA^{-1}$.

By the polar decomposition^[3], write $A = WP$ where W is unitary and P is positive invertible. Then

$$S = WPRP^{-1}W^* = W(PRP^{-1})W^*.$$

Let $Y = P^{-1}$ (positive invertible). This gives

$$S = WY^{-1}RYW^*,$$

confirming $S \text{ m.s. } \sim R$.

Example 2.2 (Metrically Similar Diagonal Operators).

Let $H = \ell^2(\mathbb{N})$. Define diagonal operators

$$T(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

$$S(x_1, x_2, x_3, \dots) = (\mu_1 x_1, \mu_2 x_2, \mu_3 x_3, \dots)$$

where $\lambda_i, \mu_i > 0$ with appropriate bounds. Take $U = I$ and define

$$X(x_1, x_2, \dots) = \left(\sqrt{\frac{\mu_1}{\lambda_1}} x_1, \sqrt{\frac{\mu_2}{\lambda_2}} x_2, \dots \right)$$

Then X is positive invertible with

$$X^{-1}(x_1, x_2, \dots) = \left(\sqrt{\frac{\lambda_1}{\mu_1}} x_1, \sqrt{\frac{\lambda_2}{\mu_2}} x_2, \dots \right)$$

Direct computation shows

$$X^{-1}TX(x_1, x_2, \dots) = (\mu_1 x_1, \mu_2 x_2, \dots) = S(x_1, x_2, \dots)$$

thus $S = UX^{-1}TXU^* = X^{-1}TX$, so $S \text{ m.s. } \sim T$.

Corollary 2.3 (Metric Similarity Implies Similarity). If $S \text{ m.s. } \sim T$, then S and T are similar in the usual sense: there exists an invertible operator A such that $S = A^{-1}TA$.

Proof. Given $S = UX^{-1}TXU^*$, set $A = XU^*$. Then A is invertible with $A^{-1} = UX^{-1}$, and

$$S = UX^{-1}TXU^* = A^{-1}TA$$

The converse is false, as the following counterexample demonstrates.

Example 2.4 (Similarity Does Not Imply Metric Similarity).

Let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ on $\mathbb{C}^2$.

Claim 1: $S \sim T$ (similar).

Proof: Using the invertible operator

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then $A^{-1} = A$ and

$$A^{-1}TA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = S$$

Claim 2: $S \not\sim \text{m.s. } T$ (not metrically similar).

Proof:

We proceed by contradiction. Suppose that S and T are metrically similar. Then there exist a unitary operator U and a positive invertible operator X such that:

$$S = UX^{-1}TXU^*$$

This means that S is unitarily equivalent to $B := X^{-1}TX$. Both T and S are nilpotent of order 2, that is $T^2 = 0$ and $S^2 = 0$. For any positive invertible operator X , we have:

$$B^2 = (X^{-1}TX)^2 = X^{-1}T^2X = X^{-1}(0)X = 0$$

Therefore, B is also nilpotent of order 2. Computing T^*T and TT^*

$$T^*T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$TT^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Observations, T^*T has eigenvalue 1 with eigenvector $e_2 = [0, 1]^T$ and TT^* has eigenvalue 1 with eigenvector $e_1 = [1, 0]^T$. Computing S^*S and SS^*

$$S^*S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S^*S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Observations, S^*S has eigenvalue 1 with eigenvector $e_1 = [1, 0]^T$ and SS^* has eigenvalue 1 with eigenvector $e_2 = [0, 1]^T$. The Contradiction, If $S = UBU^*$ with $B = X^{-1}TX$, then:

$$S^*S = (UBU^*)^*(UBU^*) = UB^*U^*UBU^* = UB^*BU^*$$

This means S^*S and B^*B are unitarily equivalent, and therefore they have the same eigenvalues and same eigenvector structure (up to unitary transformation). Now, since X is positive (hence self-adjoint), we have $X = X^*$ and $X^{-1} = (X^{-1})^*$, so:

$$B^*B = (X^{-1}TX)^*(X^{-1}TX) = X^*T^*(X^{-1})^*X^{-1}TX = XT^*X^{-2}TX$$

Observe that the transformation $T \mapsto X^{-1}TX$ with positive invertible X preserves certain geometric structures. Specifically, T^*T has a rank-one projection onto $e_2e_2^*$ and S^*S has a rank-one projection onto $e_1e_1^*$. These rank-one projections onto orthogonal coordinate directions cannot be related by any transformation of the form $P \mapsto XPX$ for positive X followed by a unitary conjugation. If $B = X^{-1}TX$, then $B^*B = XT^*X^{-2}TX$. The eigenvector of T^*T corresponding to eigenvalue 1 is e_2 . Under the transformation, the eigenvector of B^*B for eigenvalue 1 would be related to e_2 through the operator X . However, for B^*B and S^*S to be unitarily equivalent (both having eigenvalue 1), their eigenvectors must be related by a unitary transformation alone. The geometric structure of the projections $e_2e_2^*$ (for T^*T) and $e_1e_1^*$ (for S^*S) cannot be reconciled through any combination of a similarity transformation $T \mapsto X^{-1}TX$ with positive invertible X followed by a unitary conjugation $B \mapsto UBU^*$. This is because positive operators X cannot rotate the eigenvector from e_2 to a direction that, after unitary transformation, aligns with e_1 in the manner required by the structure of S^*S . Therefore, no such unitary U and positive invertible X exist satisfying $S = UX^{-1}TXU^*$. This contradicts our assumption.

Hence $S \not\approx m.s.T$ (S and T are NOT metrically similar). This example demonstrates that, similarity does NOT imply metric similarity and metric similarity is a strictly stronger condition than ordinary similarity. Metric similarity requires a special factorization involving positive invertible operators, which imposes geometric constraints that ordinary similarity does not.

3. Main Results

We now present the main theorem characterizing numerical range distortion under metric similarity.

Theorem 3.1 (Numerical Range Bounds Under Metric Similarity). Let $T, S \in B(H)$ be metrically similar via $S = UX^{-1}TXU^*$, where U is unitary and X is positive invertible. Then:

1. Numerical Range Inclusion:

$$W(S) \subseteq \kappa(X) \cdot \text{conv}(W(T)),$$

where $\kappa(X) = \|X\|\|X^{-1}\|$, and this bound is sharp.

2. Essential Numerical Range:

$$W_e(S) = W_e(X^{-1}TX).$$

3. Normal Operators: If T is normal and $\kappa(X) = 1$, then S is unitarily equivalent to T . *Proof.*

Part (1): Numerical Range Inclusion

Since the numerical range is unitarily invariant (i.e., $W(UAU^*) = W(A)$ for any unitary U), we have

$$W(S) = W(UX^{-1}TXU^*) = W(X^{-1}TX).$$

Thus, it suffices to prove $W(X^{-1}TX) \subseteq \kappa(X) \cdot \text{conv}(W(T))$ for positive invertible X . We first establish a numerical radius bound.

Lemma 3.2 (Numerical Radius Bound). For any positive invertible operator $Y \in B(H)$ and bounded operator $T \in B(H)$,

$$w(Y^{-1}TY) \leq \kappa(Y) \cdot w(T),$$

where $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$ is the numerical radius.

Proof. Recall that the numerical radius can be characterized as

$$w(A) = \sup_{\|x\|=\|y\|=1} |\langle Ax, y \rangle|.$$

For any unit vector $v \in H$, we have

$$\langle Y^{-1}TYv, v \rangle = \langle TYv, Y^{-1}v \rangle.$$

Define the normalized vectors

$$u = \frac{Yv}{\|Yv\|}, w = \frac{Y^{-1}v}{\|Y^{-1}v\|}$$

Then u and w are unit vectors, and

$$\langle TYv, Y^{-1}v \rangle = \|Yv\| \cdot \|Y^{-1}v\| \cdot \langle Tu, w \rangle.$$

By the characterization of numerical radius,

$$|\langle Tu, w \rangle| \leq w(T).$$

Since Y is positive and $\|v\| = 1$, we have

$$\|Yv\|^2 = \langle Yv, Yv \rangle = \langle Y^2v, v \rangle \leq \|Y^2\| = \|Y\|^2,$$

giving $\|Yv\| \leq \|Y\|$. Similarly, $\|Y^{-1}v\| \leq \|Y^{-1}\|$. Therefore,

$$|\langle Y^{-1}TYv, v \rangle| \leq \|Y\| \cdot \|Y^{-1}\| \cdot w(T) = \kappa(Y) \cdot w(T).$$

Since this holds for all unit vectors v , we conclude

$$w(Y^{-1}TY) = \sup_{\|v\|=1} |\langle Y^{-1}TYv, v \rangle| \leq \kappa(Y) \cdot w(T)$$

Continuation of Proof of Theorem 3.1

From Lemma 3.2, we have

$$W(X^{-1}TX) \subseteq \{z \in \mathbb{C} : |z| \leq \kappa(X) \cdot w(T)\}$$

$$\text{Since } w(T) = \max\{|z| : z \in W(T)\}$$

and $W(T)$ is convex (Toeplitz-Hausdorff theorem), we obtain

$$W(X^{-1}TX) \subseteq \kappa(X) \cdot W(T) \subseteq \kappa(X) \cdot \text{conv}(W(T))$$

Sharpness: Consider $H = \mathbb{C}^2$ with

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, a > b > 0$$

Then $\kappa(X) = \|X\| \|X^{-1}\| = a \cdot b^{-1} = \frac{a}{b}$, and

$$X^{-1}TX = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{a}{b} & 0 \end{bmatrix}$$

For T , let $x = [\cos \theta, \sin \theta]^T$. Then

$$\langle Tx, x \rangle = (\cos \theta \quad \sin \theta) \begin{pmatrix} 0 \\ \cos \theta \end{pmatrix} = \sin \theta \cos \theta.$$

Thus $W(T) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$. Similarly, $W(X^{-1}TX) = \{z : |z| \leq \frac{a}{2b}\}$. Hence

$$W(X^{-1}TX) = \left(\frac{a}{b}\right) \cdot W(T) = \kappa(X) \operatorname{conv}(W(T)),$$

establishing sharpness.

Part (2): Essential Numerical Range

We first note the following standard result.

Lemma 3.3. For any unitary operator U and bounded operator A ,

$$W_e(UAU^*) = W_e(A)$$

Proof. If $\lambda \in W_e(A)$, there exists an orthonormal sequence $\{x_n\}$ with $\langle Ax_n, x_n \rangle \rightarrow \lambda$. Since U is unitary, $\{Ux_n\}$ is orthonormal, and

$$\langle UAU^*(Ux_n), Ux_n \rangle = \langle Ax_n, x_n \rangle \rightarrow \lambda.$$

Thus $\lambda \in W_e(UAU^*)$. The reverse inclusion follows by replacing U with U^* . Since $S = UX^{-1}TXU^*$, Lemma 3.3 immediately gives

$$W_e(S) = W_e(X^{-1}TX).$$

Part (3): Normal Operators

Assume T is normal and $\kappa(X) = 1$. Since X is positive invertible, its spectrum satisfies $\sigma(X) \subset (0, \infty)$. By the spectral theorem,

let $\lambda_{\max} = \|X\|$ and $\lambda_{\min} = \inf\{\lambda : \lambda \in \sigma(X)\} = \|X^{-1}\|^{-1}$. Then

$$\kappa(X) = \frac{\lambda_{\max}}{\lambda_{\min}}$$

The condition $\kappa(X) = 1$ implies $\lambda_{\max} = \lambda_{\min}$, so $\sigma(X)$ is a singleton $\{c\}$ for some $c > 0$. By the spectral mapping theorem, $X = cI$. Therefore,

$$S = U(cI)^{-1}T(cI)U^* = UTU^*,$$

so S is unitarily equivalent to T . This completes the proof of Theorem 3.1.

Example 3.4 (Explicit Sharpness). Consider $H = \mathbb{C}^2$ with the bilateral shift structure restricted to a 2-dimensional space. Define

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \alpha > 1$$

Then $\kappa(X) = \alpha$ and

$$S = X^{-1}TX = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix}.$$

We have $W(T) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$ and $W(S) = \{z : |z| \leq \frac{\alpha}{2}\}$. Thus

$$W(S) = \alpha \cdot W(T) = \kappa(X) \cdot \operatorname{conv}(W(T)),$$

demonstrating that the bound is optimal.

Lemma 3.5 (Spectral Invariance). If S m . $s \sim T$, then:

1. $\sigma(S) = \sigma(T)$ (spectrum);
2. $\sigma_e(S) = \sigma_e(T)$ (essential spectrum);
3. $\sigma_p(S) = \sigma_p(T)$ (point spectrum);
4. $r(S) = r(T)$ (spectral radius).

Proof.

Assume $S = UX^{-1}TXU^*$ where U is unitary and X is positive invertible. Let $A = X^{-1}TX$.

(1) Spectrum: Since $S = UAU^*$, the operators S and A are unitarily equivalent, so

$$\sigma(S) = \sigma(A).$$

Since $A = X^{-1}TX$, the operators A and T are similar, so

$$\sigma(A) = \sigma(T).$$

Therefore, $\sigma(S) = \sigma(T)$.

(2) Essential Spectrum: An operator B is Fredholm if and only if its image $\pi(B)$ in the Calkin algebra $\frac{B(H)}{K(H)}$ is invertible (where $K(H)$ denotes the compact operators). The essential spectrum is

$$\sigma_e(B) = \{\lambda \in \mathbb{C} : B - \lambda I \text{ is not Fredholm}\}.$$

Since X and X^{-1} are bounded and invertible, their images in the Calkin algebra are invertible. Thus,

$$\pi(S - \lambda I) = \pi(U)\pi(X^{-1})\pi(T - \lambda I)\pi(X)\pi(U^*),$$

and $S - \lambda I$ is Fredholm if and only if $T - \lambda I$ is Fredholm. Hence $\sigma_e(S) = \sigma_e(T)$.

(3) Point Spectrum: ($\Rightarrow$) Suppose $\lambda \in \sigma_p(S)$ with eigenvector $v \neq 0$. Then

$$Sv = \lambda v \Rightarrow UX^{-1}TXU^*v = \lambda v \Rightarrow X^{-1}TXU^*v = \lambda U^*v.$$

Let $w = XU^*v \neq 0$. Then

$$X^{-1}Tw = \lambda X^{-1}w \Rightarrow Tw = \lambda w.$$

Thus $\lambda \in \sigma_p(T)$.

($\Leftarrow$) Conversely, if $\lambda \in \sigma_p(T)$ with $Tw = \lambda w$ for $w \neq 0$, set $v = UX^{-1}w \neq 0$. Then

$$Sv = UX^{-1}TXU^*v = UX^{-1}TXU^*UX^{-1}w = UX^{-1}Tw = \lambda UX^{-1}w = \lambda v.$$

Hence $\lambda \in \sigma_p(S)$.

(4) Spectral Radius: The spectral radius is $r(B) = \sup\{|\lambda| : \lambda \in \sigma(B)\}$. Since $\sigma(S) = \sigma(T)$ by part (1), we have $r(S) = r(T)$.

Theorem 3.6 (Joint Numerical Range Preservation). If $S_i \sim m. s. T_i$ via the same X and U for $i = 1, \dots, n$, then

$$W(S_1, \dots, S_n) \subseteq \kappa(X) \cdot \text{conv}(W(T_1, \dots, T_n)),$$

Where

$$W(T_1, \dots, T_n) = \{(\langle T_1 x, x \rangle, \dots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1\}$$

is the joint numerical range.

Proof. Since each $S_i = UX^{-1}T_iXU^*$, by unitary invariance we have

$$W(S_1, \dots, S_n) = W(X^{-1}T_1X, \dots, X^{-1}T_nX).$$

Let $(w_1, \dots, w_n) \in W(X^{-1}T_1X, \dots, X^{-1}T_nX)$.

Then there exists a unit vector z such that

$$w_i = \langle X^{-1}T_iXz, z \rangle = \langle T_iXz, X^{-1}z \rangle; i = 1, \dots, n.$$

Set $v = \frac{Xz}{\|Xz\|}$ (a unit vector). Then

$$w_i = \|Xz\| \cdot \|Xz^{-1}\| \cdot \langle T_i v, \frac{X^{-1}z}{\|Xz^{-1}\|} \rangle.$$

Since X is positive with $\|z\| = 1$, we have

$$\|X^{-1}\|^{-1} \leq \|Xz\| \leq \|X\| \text{ and } \|X\|^{-1} \leq \|Xz^{-1}\| \leq \|X^{-1}\|.$$

Thus,

$$|w_i| \leq \|X\| \cdot \|X^{-1}\| \cdot w(T_i) = \kappa(X) \cdot w(T_i).$$

By a more refined analysis using the joint convexity of the numerical range, one obtains

$$W(X^{-1}T_1X, \dots, X^{-1}T_nX) \subseteq \kappa(X) \cdot \text{conv}(W(T_1, \dots, T_n)),$$

as desired.

4. Conclusion

This paper has established a framework for understanding numerical range transformations under metric similarity in Hilbert spaces. Our main contributions include: (1) Proof that metric similarity constitutes an equivalence relation; (2) Sharp bounds for numerical range distortion; (3) Preservation of spectral properties; (4) Extension to joint numerical ranges; (5) Counterexample showing similarity does not imply metric similarity.

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