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Existence of at least one solution of Volterra integro-differential equation with nonlocal condition

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Abstract

In this work, we study the existence of at least one absolutely continuous solution of an initial value problem of integro-differential equation with nonlocal condition.

Keywords: Volterra, integro-differential, nonlocal condition

Introduction

Mathematical modelling of real-life problems usually results in partial differential equations or integral and integro-differential equations, stochastic equations. Many mathematical formulation of physical phenomena contain integro-differential equations, these equations arises in many fields like fluid dynamics, biological models and chemical kinetics integrodifferential equations are usually difficult to solve analytically so it is required to obtain an efficient approximate solution.

Here we are concerned with the nonlinear functional integral equation

$$\frac{dx}{dt} = a(t) + \int_0^t g\left(t, s, \frac{dx}{dt}\right) ds, a.e, t \in (0, T) \quad (1)$$

With the nonlocal condition,

$$x(0) + \sum_{k=1}^n a_k(\tau_k) = x_0, \tau_k \in (0, T) \quad (2)$$

We study the existence of at least one absolutely continuous solution of an initial value problem of integro-differential equation (1) with nonlocal condition (2).

To achieve our goal we study the existence of at least one integrable solution of Volterra integro-differential equation

$$y(t) = a(t) + \int_0^t g(t, s, y(s)) ds, a.e, t \in (0, T) \quad (3)$$

The main result will be based on the following theorems.

Theorem 1.1 (Lebesgue Dominated Convergence Theorem) ^[7].

Let $\{f_n\}$ be a sequence of functions converging to a limit f on A , and suppose that

$$|f_n(t)| \leq \phi(t), t \in A, n = 1, 2, \dots,$$

Where ϕ is an integrable function on A . Then f is integrable on A and

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$$\lim_{n \rightarrow \infty} \int_A f_n(t) d\mu = \int_A f(t)$$

Theorem 1.2 (Kolmogorov compactness theorem) ^[7].

Let $\Omega \subset L_p(0,1)$, $1 \leq p < \infty$ if Ω is bounded in $L_p(0,1)$ and $x_h \rightarrow x$ as $h \rightarrow 0$ uniformly with respect to $x \in \Omega$ where

$$x_h = \frac{1}{h} \int_t^{t+h} x(\theta) d\theta \text{ then } (\Omega) \text{ is relatively compact in } L^p(0,1).$$

Theorem 1.3 (Schauder fixed point Theorem) ^[6].

Let U be a convex subset of a Banach space X , and $T: U \rightarrow U$ is compact, continuous map. Then T has at least one fixed point in U .

Integral representation

Lemma 2.1 Let the set of the problem (1) and (2) be exists, then it can be expressed by the functional integral equation

$$x(t) = \frac{1}{1 - \sum_{k=1}^n a_k} (x_0 + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds) + \int_0^t y(s) ds \quad (4)$$

Proof.

Let $\frac{dx}{dt} = y \in C[0, T]$, then

$$y(t) = a(t) + \int_0^t f(t, s, y(s)) ds$$

Where;

$$x(t) = x(0) + \int_0^t y(s) ds \quad (5)$$

Using the nonlocal condition (2), we obtain

$$x(\tau_k) = x(0) + \int_0^{\tau_k} y(s) ds \text{ and } a_k x(\tau_k) = a_k x(0) + a_k \int_0^{\tau_k} y(s) ds,$$

$$\sum_{k=1}^n a_k x(\tau_k) = \sum_{k=1}^n a_k x(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds \text{ and}$$

$$x(0) - x_0 = \sum_{k=1}^n a_k x(0) + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds, \quad \text{then} \quad (1 - \sum_{k=1}^n a_k) x(0) = x_0 + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds.$$

Hence

$$x(0) = \frac{1}{1 - \sum_{k=1}^n a_k} (x_0 + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds)$$

Substituting in (5), we obtain

$$x(t) = \frac{1}{1 - \sum_{k=1}^n a_k} (x_0 + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds) + \int_0^t y(s) ds.$$

Existence of integrable solution

Consider the functional integral equation (3) under the following assumptions

1. $g: [0, T] \times [0, T] \times R \rightarrow R$ is continuous in $t \in [0, T]$ and in $x \in R_+$ for all $s \in [0, T]$ and measurable in $s \in [0, T]$ for any $x \in R_+$.
2. There exists an integrable function k and positive constant L such that

$$|g(t, s, x)| \leq |k(t, s)| + L|x| \text{ and}$$

$$\int_0^T \int_0^t |k(t, s)| ds dt \leq k^*.$$

3. $a: [0, T] \rightarrow [0, T]$ is integrable and $\int_0^T |a(t)| dt \leq M$, M is positive constant.

Now for the existence of at least one integrable solution of the functional integral equation

4. We have the following theorem.

Theorem (3.1)

Let the assumptions (1)-(3) be satisfied, then the functional integral equation (3) has at least one integrable solution on $y \in L^1[0, T]$.

Proof. Define the operator G by

$$Gy(t) = a(t) + \int_0^t g(t, s, y(s)) ds.$$

Set $Q_r = \{y \in R: |y| \leq r\}$. Let $y \in Q_r$, now we will prove that $\{Gy\}$ is uniformly bounded.

$$\begin{aligned} |Gy(t)| &= |a(t) + \int_0^t g(t, s, y(s)) ds| \\ &\leq |a(t)| + \int_0^t |g(t, s, y(s))| ds \\ &\leq |a(t)| + \int_0^t [|k(t, s)| + L|y(s)|] ds \\ &\leq |a(t)| + \int_0^t |k(t, s)| ds + L \int_0^t |y(s)| ds, \end{aligned}$$

Then

$$\begin{aligned} \|Gy(t)\| &= \int_0^T |Gy(t)| dt \leq M + k^* + L \int_0^t \|y\| ds \\ &\leq M + k^* + L \int_0^t r ds \\ &\leq M + k^* + LTr = r. \end{aligned}$$

Now to prove that G is compact, we apply Kolmogorov Theorem (1.2).

Let Ω be bounded subset of Q_r and $G: Q_r \rightarrow Q_r$ then $G(\Omega)$ is also bounded in $L^1[0, T]$.

Now we will prove that $(Gy_h) \rightarrow (Gy)$ uniform in $L^1[0, T]$. Let $y \in \Omega$, then

$$\begin{aligned} \|Gy_h - Gy\| &= \int_0^T |(Gy_h)(t) - (Gy)(t)| dt \\ &= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Gy)(\theta) d\theta - (Gy)(t) \right| dt \\ &\leq \int_0^T \frac{1}{h} \int_t^{t+h} |(Gy)(\theta) - (Gy)(t)| d\theta dt \\ &\leq \int_0^T \frac{1}{h} \int_t^{t+h} \left| [a(\theta) + \int_0^t g(\theta, s, y(s)) ds] - [a(t) + \int_0^t g(t, s, y(s)) ds] \right| d\theta dt. \end{aligned}$$

Now $g \in L^1[0, T]$, it follows that

$$\frac{1}{h} \int_t^{t+h} \left| [a(\theta) + \int_0^t g(\theta, s, y(s)) ds] - [a(t) + \int_0^t g(t, s, y(s)) ds] \right| d\theta dt \rightarrow 0$$

as $h \rightarrow 0$, then $(Gy_h) \rightarrow (Gy)$ uniformly, hence $G(\Omega)$ is relatively compact.

Then G is compact operator. Now we will prove that G is continuous operator.

Let $\{y_n\} \subset Q_r, y_n \rightarrow y$

$$Gy_n(t) = a(t) + \int_0^t g(t, s, y_n(s)) ds$$

and

$$\lim_{n \rightarrow \infty} Gy_n(t) = a(t) + \lim_{n \rightarrow \infty} \int_0^t g(t, s, y_n(s)) ds$$

Now g is continuous in y , $y_n \rightarrow y$. Then by using Lebeague dominated convergence theorem (1.1), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Gy_n(t) &= a(t) + \int_0^t \lim_{n \rightarrow \infty} g(t, s, y_n(s)) ds \\ &= a(t) + \int_0^t g(t, s, y(s)) ds \\ &= Gy(t). \end{aligned}$$

This means that the operator G is continuous and by the Schauder fixed point theorem (1.3) there exists at least one solution on $L^1[0, T]$ of the functional integral equation (3).

Initial value problem

Consider now the initial value problem (1).

$$\frac{dx}{dt} = a(t) + \int_0^t f(t, s, \frac{dx}{ds}) ds, \quad t \in (0, T]$$

with the nonlocal condition (2).

$$x(0) + \sum_{k=1}^n a_k x(\tau_k) = x_0, \quad \tau_k \in (0, T)$$

For existence of absolutely continuous solution of (1), we have the following theorem.

Theorem 4.1 Let the assumptions of Theorem (3.1) be satisfied, then there exists at least one solution $x \in AC[0, T]$ of the initial value problem (1) and (2).

Proof. The solution of problem (1) and (2) is given by

$$x(t) = \frac{1}{1 - \sum_{k=1}^n a_k} (x_0 + \sum_{k=1}^n a_k \int_0^{\tau_k} y(s) ds) + \int_0^t y(s) ds \in AC[0, T]$$

$$\text{Where; } y(t) = a(t) + \int_0^t g(t, s, y(s)) ds \in L^1[0, T]$$

Then from Theorem (2.1) we deduce that there exists at least one solution of the problem (1) and (2).

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