

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2025; 10(8): 17-22
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 06-05-2025
Accepted: 07-06-2025

FN Mburu
School of Pure and Applied
Sciences, Kirinyaga University,
Kenya

PW Njori
School of Pure and Applied
Sciences, Kirinyaga University,
Kenya

John Wahome
Department of Mathematics,
School of Science, Laikipia
University, Kenya

Study on W_7 -curvature tensor within the framework of Lorentzian para-Kenmotsu Manifolds

FN Mburu, PW Njori and John Wahome

DOI: <https://www.doi.org/10.22271/math.2025.v10.i8a.2113>

Abstract

This paper discusses the properties of W_7 -curvature tensors on L P Kenmotsu manifolds. Properties of this Curvature tensor under various conditions on these manifolds are explored and their geometric implications are examined. The study includes investigations of W_7 -flatness, ξ - W_7 -flatness, ϕ - W_7 flatness, W_7 -Semisymmetric, $W_7 \cdot R = 0$.

Keywords: Lorentzian Para-Kenmotsu, W_i curvature tensor, paper discusses, riemannian manifold

1. Introduction

Abdul Haseeb and Rajendra Prasad in 2018 introduced Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifolds, studying ϕ -semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons. 1970, [5] Pokhariyal and Mishra introduced new tensor fields on a Riemannian manifold, called the Weyl-projective curvature tensor of type (1, 3) and the tensor field E. The concept of W_7 -curvature tensor was defined by Tripathi and Gupta [15] of an n (where $n=2m+1$)-dimensional Riemannian manifold are, respectively, defined as

$$W_7(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1} [g(Y, Z)S(X, T) - g(X, T)S(Y, Z)] \quad (1)$$

$$W_7(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [g(Y, Z)QX - S(Y, Z)X]$$

For all $(X, Y, Z) \in_X(M)$. Where R represents the curvature tensor and S corresponds to the Ricci tensor of the manifold.

2. Preliminaries

A differentiable manifold M admitting a (ϕ, ξ, η, g) , (1,1) tensor field ϕ , contravariant vector field ξ , a 1-form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold [7] if it satisfies:

$$\phi^2 X = X + \eta(X)\xi \quad (2)$$

$$\eta(\xi) = -1 \quad (3)$$

$$\phi\xi = 0 \quad (4)$$

$$\text{rank}\phi = n-1 \quad (5)$$

$$\phi(X, Y) = \phi(Y, X) \quad (6)$$

Where,

$$\phi(X, Y) = g(X, \phi Y) \quad (7)$$

Corresponding Author:
FN Mburu
School of Pure and Applied
Sciences, Kirinyaga University,
Kenya

In the Lorentzian Para Kenmotsu manifold we have

$$g(X, \phi Y) = g(\phi X, Y) \quad (8)$$

$$g(X, \xi) = \eta(X) \quad (9)$$

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold (2018) [3] if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (10)$$

On the LP-Kenmotsu manifold, the following relations hold [4]

$$\nabla_X \xi = -\phi^2 X - X - \eta(X)\xi \quad (11)$$

$$(\nabla_X \eta)Y = -g(X, Y)\eta(X)\eta(Y) \quad (12)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \quad (13)$$

$$R(\xi, X)\xi = X + \eta(X)\xi = -\nabla_X \xi, \quad (14)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (15)$$

$$S(X, \xi) = (n-1)\eta(X), \quad (16)$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (17)$$

$$Q\xi = (n-1)\xi \quad (18)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \quad (19)$$

For vector fields X, Y, Z on M and where S is Ricci Tensor and Q, Ricci operator and R Curvature tensor with respect to Levi-Civita connection ∇.

A Lorentzian Para-Kenmotsu manifold M is said to be an η -Einstein manifold if its Ricci-tensor $S(X, Y)$ is of the form (2021) [7]

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \quad (20)$$

Where a and b are scalar functions on M . In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

3. A W_7 -flat L.P Kenmotsu manifold

Definition 3.1 An n -dimensional L. P. Kenmotsu manifold is said to be W_7 -flat if its W_7 -curvature tensor satisfies the following condition

$$W_7(X, Y)Z = 0$$

Theorem 3.1; W_7 -flat L.P-Kenmotsu manifold is a flat manifold.

Proof.

Suppose the LP-Kenmotsu manifold is W_7 -flat, then the following hold

$$W_7(X, Y)Z = 0$$

$$W_7(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [g(Y, Z)QX - S(Y, Z)X] = 0 \quad (3a)$$

$$(n-1)R(X, Y)Z = S(Y, Z)X - (n-1)(g(Y, Z)X)$$

$$(n-1)R(X, Y)Z = (n-1)(g(Y, Z)X - (n-1)(g(Y, Z)X) \quad (3b)$$

Equation (3b) reduces to

$$(n-1)R(X, Y)Z = 0 \quad (3c)$$

Equation (3c) yields

$$R(X, Y)Z = 0 \quad (3d)$$

Hence proved.

4. A ξ - W_7 -flat Lorentzian Para Kenmotsu manifold

Definition 4.1: An n -dimensional LP-Kenmotsu manifold is said to be ξ - W_7 -flat if its

W_7 -curvature tensor satisfies the following condition.

$$W_7(X, Y)\xi = 0$$

Theorem 4.1: An ξ - W_7 flat Lorentzian Para Kenmotsu manifold is a special type of η -Einstein manifold.

Let us consider

$$W_7(X, Y)\xi = R(X, Y)\xi + \frac{1}{n-1} \{g(Y, \xi)QX - S(Y, \xi)X\} \quad (4a)$$

$$0 = \eta(Y)X - \eta(X)Y + g(Y, \xi)X - g(Y, \xi)X \quad (4b)$$

But

$$S(Y, \xi)X = (n-1)g(Y, \xi)X$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

$$\eta(X)Y = \eta(Y)X \quad (4c)$$

$$\eta(X)g(Y, U) = \eta(Y)g(X, U) \quad (4d)$$

Setting $Y = \xi$

$$\eta(X)g(\xi, U) = \eta(\xi)g(X, U) \quad (4e)$$

$$\eta(X)\eta(U) = -g(X, U)$$

$$g(X, U) = -\eta(X)\eta(U)$$

$$S(X, U) = -(n-1)\eta(X)\eta(U) \quad (4f)$$

Hence (4f) is a special η -Einstein manifold

5. W_7 -Semisymmetric Lorentzian Para Kenmotsu manifold

Definition 5.1 An n -dimensional W_7 -Lorentzian-Para-Kenmotsu manifold M is called Semisymmetric if for all vector fields X, Y, Z, U, V on M , the following holds:

$$(R(X, Y) \cdot R)(U, V)Z = 0. \quad (5a)$$

Where $R(X, Y)$ serves as a derivation on the curvature tensor W_7 .

Theorem 5.1 Any Lorentzian Para-Kenmotsu manifold is semisymmetric.

Proof of Theorems

Proof of Theorem 5.1

Proving that

$$(R(X, Y) \cdot R)(U, V)Z = 0. \quad (5b)$$

Hence, expanding $(R(X, Y) \cdot R)(U, V)Z$ we get

$$(R(X, Y) \cdot R)(U, V)Z = R(X, Y)R(U, V)Z - R(R(X, Y)U, V)Z - R(U, R(X, Y)V)Z - R(U, V)R(X, Y)Z \quad (5c)$$

Putting $U = \xi$ in 5.c above gives

$$(R(X, Y) \cdot R)(\xi, V)Z = R(X, Y)R(\xi, V)Z - R(R(X, Y)\xi, V)Z - R(\xi, R(X, Y)V)Z - R(\xi, V)R(X, Y)Z \quad (5d)$$

Simplifying each of the four terms in (5d) separately yields:

$$\text{First term: } R(X, Y)R(\xi, V)Z$$

$$R(X, Y)R(\xi, V)Z = R(X, Y)[g(V, Z)\xi - \eta(Z)V]$$

$$R(X, Y)R(\xi, V)Z = [g(Y, g(V, Z)\xi)X - g(X, g(V, Z)\xi)Y - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y$$

$$R(X, Y)R(\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5e)$$

Second Term: $R(R(X, Y)\xi, V)Z$

$$R(R(X, Y)\xi, V)Z = R([\eta(Y)X - \eta(X)Y], V)Z$$

$$= g(V, Z)\eta(Y)X - g(V, Z)\eta(X)Y - g([\eta(Y)X - \eta(X)Y], Z)V$$

$$\therefore R(R(X, Y)\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Y)g(X, Z)V + \eta(X)g(Y, Z)V \quad (5f)$$

Third term: $R(\xi, R(X, Y)V)Z$

$$R(\xi, R(X, Y)V)Z = R(\xi, [g(Y, V)X - g(X, V)Y])Z$$

$$= g([g(Y, V)X - g(X, V)Y], Z)\xi - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y$$

$$\therefore R(\xi, R(X, Y)V)Z = g(X, Z)g(Y, V)\xi - g(Y, Z)g(X, V)\xi - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5g)$$

Fourth term: $R(\xi, V)R(X, Y)Z$

$$R(\xi, V)R(X, Y)Z = g(V, R(X, Y)Z)\xi - \eta(R(X, Y)Z)V$$

$$= g(V, [g(Y, Z)X - g(X, Z)Y])\xi - \eta([g(Y, Z)X - g(X, Z)Y])V$$

$$\therefore R(\xi, V)R(X, Y)Z = g(V, X)g(Y, Z)\xi - g(V, Y)g(X, Z)\xi - \eta(X)g(Y, Z)V + \eta(Y)g(X, Z)V \quad (5h)$$

Putting equations (5e), (5f), (5g) and (5h) in equation 5d) gives

$$\therefore (R(X, Y) \cdot R)(\xi, V)Z = 0 \quad (5.i)$$

This completes the proof that Lorentzian Para-Kenmotsu is a semisymmetric manifold.

6. W_7 – Para Kenmotsu Manifold satisfying the condition $W_7 \cdot R = 0$

Definition 6.1 A Lorentzian-Para-Kenmotsu manifold M is said to satisfy $W_7 \cdot R = 0$ condition if W_7

$$(W_7(U, V) \cdot R)(X, Y)Z = 0 \quad (6a)$$

For any vector fields X, Y, Z, U, V on M .

Theorem 6.1, A W_7 -Lorentzian-para-Kenmotsu manifold satisfies the condition $W_7 \cdot R = 0$

Proof of theorem 6.1

Proving that the relation (6a) holds for a W_7 -Lorentzian para-Kenmotsu Manifold

The above equation (6.a) can be written as follows:

$$(W_7(U, V) \cdot R)(X, Y)Z = W_7(U, V)R(X, Y)Z - R(W_7(U, V)X, Y)Z - R(X, W_7(U, V)Y)Z$$

$$- R(X, Y)W_7(U, V)Z \quad (6.b)$$

Putting $U = \xi$ in (6.2) gives

$$(W_7(\xi, V) \cdot R)(X, Y)Z = W_7(\xi, V)R(X, Y)Z - R(W_7(\xi, V)X, Y)Z - R(X, W_7(\xi, V)Y)Z - R(X, Y)W_7(\xi, V)Z \quad (6c)$$

But

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y$$

$$W_7(\xi, V)W = g(V, W)\xi - g(\xi, W)V$$

$$W_7(\xi, V)W = g(V, W)\xi - \eta(W)V \quad (6d)$$

Computing the four terms in (6c) separately gives

First term: $W_7(\xi, V)R(X, Y)Z$

From (6e), gives

$$W_7(\xi, V)W = [g(V, [g(Y, Z)X - g(X, Z)Y])\xi - \eta([g(Y, Z)X - g(X, Z)Y])V]$$

$$W_7(\xi, V)R(X, Y)Z = g(V, X)g(Y, Z)\xi - g(V, Y)g(X, Z)\xi - \eta(X)g(Y, Z)V - \eta(Y)g(X, Z)V \quad (6f)$$

Second Term: $R(W_7(\xi, V)X, Y)Z$

$$R([g(V, X)\xi - g(\xi, X)V], Y)Z = g(Y, Z)[g(V, X)\xi - g(\xi, X)V] - g([g(V, X)\xi - g(\xi, X)V], Z)Y$$

$$R([g(V, X)\xi - g(\xi, X)V], Y)Z = g(Y, Z)g(V, X)\xi - g(Y, Z)g(\xi, X)V - g(\xi, Z)g(V, X)Y + g(V, Z)g(\xi, X)Y$$

$$W_7(\xi, V)R(X, Y)Z = g(Y, Z)g(V, X)\xi - \eta(X)g(Y, Z)V - \eta(Z)g(V, X)Y + \eta(X)g(V, Z)Y \quad (6g)$$

Third Term: $R(X, W_7(\xi, V)Y)Z$

$$R(X, W_7(\xi, V)Y)Z = g(W_7(\xi, V)Y, Z)X - g(X, Z)W_7(\xi, V)Y$$

$$R(X, W_7(\xi, V)Y)Z = g([g(V, Y)\xi - \eta(Y)V], Z)X - g(X, Z)[g(V, Y)\xi - \eta(Y)V]$$

$$R(X, W_7(\xi, V)Y)Z = \eta(Z)g(V, Y)X - \eta(Y)g(V, Z)X - g(X, Z)g(V, Y)\xi + \eta(Y)g(X, Z)V \quad (6h)$$

Fourth Term: $R(X, Y)W_7(\xi, V)Z$

$$W_7(\xi, V)Z = g(V, Z)\xi - \eta(Z)V$$

$$R(X, Y)W_7(\xi, V)Z = g(Y, [g(V, Z)\xi - \eta(Z)V])X - g(X, [g(V, Z)\xi - \eta(Z)V])Y$$

$$R(X, Y)W_7(\xi, V)Z = \eta(Y)g(V, Z)X - \eta(Z)g(Y, V)X - \eta(X)g(V, Z)Y + \eta(Z)g(X, V)Y \quad (6i)$$

Plugging in equations (6e), (6g), (6h) and (6i) together gives

$$W_7 \cdot R = 0$$

This completes the proof of the theorem (6.1)

7. A ϕ - W_7 -LP-Kenmotsu manifold

Definition 7.1: A LP-Kenmotsu manifold is said to be ϕ - W_7 -flat if

W_7 -curvature tensor satisfies the following condition

$$W_7(X, Y) \cdot \phi Z = W_7(X, Y) \phi Z - \phi(W_7(X, Y)Z) = 0 \quad (7a)$$

Theorem 7: A W_7 -LP-Kenmotsu –manifold is a ϕ - W_7 flat manifold

Proof.

Consider ϕ - W_7 LP-Kenmotsu manifold, then the following hold

$$W_7(X, Y) \phi Z - \phi(W_7(X, Y)Z)$$

$$= R(X, Y) \phi Z - g(Y, Z) \phi X + g(X, Z) \phi Y + g(X, Y) \phi Z - g(Y, \phi Z) X - g(X, Y) \phi Z + g(Y, Z) \phi X \quad (7b)$$

$$= R(X, Y) \phi Z + g(X, Z) \phi Y - g(Y, \phi Z) X$$

$$R(X, Y) \phi Z = g(Y, \phi Z) X - g(X, Z) \phi Y \quad (7c)$$

$$= g(Y, \phi Z) X - g(X, \phi Z) Y - g(Y, Z) \phi X + g(X, Z) \phi Y + g(X, Y) \phi Z - g(Y, \phi Z) X - g(X, Y) \phi Z + g(Y, Z) \phi X \quad (7d)$$

$$= -g(X, \phi Z) Y + g(X, Z) \phi Y \quad (7e)$$

Let $Y=Z=\xi$ in above equation yields

$$W_7(X, \xi) \cdot \phi \xi = 0$$

This completes the proof.

References

1. Mburu F. Study of some curvature tensors on Lorentzian para-Kenmotsu manifolds; 2025.
2. Mishra RS. On Sasakian manifold. Indian J Pure Appl Math. 1972;3(5):739-749.
3. Pokhariyal GP. On symmetric Sasakian manifold. Kenya J Sci Ser A. 1988;9(1-2):39-42.
4. Pokhariyal GP, Mishra RS. Curvature tensor and their relativistic significance II. Yokohama Math J. 1971;19(2):97-103.
5. De UC, Guha N. Conharmonic recurrent Sasakian manifold. Indian J.; 1992, p. 209-215.
6. Tripathi MM, De UC. Lorentzian almost para-contact manifolds and their submanifolds. Pure Appl Math. 2001;8(2):101-125.
7. Njori PW, Moindi SK, Pokhariyal GP. A study on W_6 -curvature tensors and W_8 -curvature tensors in Kenmotsu manifolds admitting semi-symmetric metric connection. 2021;1:41-57.
8. Njori PW. Curvature tensors on semi-Riemannian and generalized Sasakian space forms admitting semi-symmetric metric connection [dissertation]. Nairobi: Univ Nairobi; 2018.
9. Satyanarayana T, Prasad KS. On semi-symmetric para-Kenmotsu manifolds. Turk J Anal Number Theory. 2015;3(6):145-148.
10. Pokhariyal GP. Relativistic significance of curvature tensors. Int J Math Math Sci. 1982;5:133-139.
11. Njori P, Mutinda F. Study of some new curvature tensors on Lorentzian para-Sasakian manifolds and other related manifolds; 2021.
12. Yoldas H, Yasar E. Some notes on Kenmotsu manifold. Facta Univ Ser Math Inform. 2021;1:949-961.
13. Devi SS, Prasad KS, Satyanarayana T. Certain curvature conditions on Lorentzian para-Kenmotsu manifolds. Reliab Theory Appl. 2022;17(2):413-421.
14. Mert T, Ateken M. On some important characterizations of Lorentz para-Kenmotsu manifolds on some special curvature tensors. Asian Eur J Math. 2024;17(1):2350246.