

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

NAAS Rating (2025): 4.49

Maths 2025; 10(7): 150-160

© 2025 Stats & Maths

<https://www.mathsjournal.com>

Received: 03-06-2025

Accepted: 07-07-2025

FN Mburu

Kirinyaga University, School of
Pure and Applied Sciences,
Kutus, Kirinyaga County,
Kenya

PW Njori

Kirinyaga University, School of
Pure and Applied Sciences,
Kutus, Kirinyaga County,
Kenya

CN Gitonga

Kirinyaga University, School of
Pure and Applied Sciences,
Kutus, Kirinyaga County,
Kenya

Analysis of some properties of w_5 curvature tensor in lorentzian para kenmotsu manifold

FN Mburu, PW Njori and CN Gitonga

DOI: <https://www.doi.org/10.22271/math.2025.v10.i7b.2109>

Abstract

This paper investigates the properties of W_5 curvature tensors on Lorentzian Para Kenmotsu manifolds. Properties of this Curvature tensor under various conditions on these manifolds are explored and their geometric implications are examined. The study also includes investigations of W_5 flatness, $\xi - W_5$ flatness, $\phi - W_5$ flatness and W_5 -Semisymmetric on Lorentzian Para Kenmotsu manifolds and their connections to η -Einstein, Einstein and special η -Einstein. Additionally, The Ricci operator's behaviors on Lorentzian Para Kenmotsu manifolds under the conditions $W_5 \cdot Q = 0$, $Q \cdot W_5 = 0$ are analyzed. Expressions for this curvature tensor while considering the condition $W_5(\xi, X) \cdot W_5 = 0$, $R \cdot W_5 = 0$ and $W_5 \cdot W_5 = 0$ are derived. Proves to determine whether these manifolds are flat will be provided. The findings of this study enhance understanding of the geometric properties of Lorentzian Para Kenmotsu manifolds in relation to W_5 curvature tensors.

Keywords: Para-contact metric manifold, Lorentzian almost Paracontact manifold, Lorentzian Para-Kenmotsu manifold, Einstein manifold, η -Einstein manifold, W_5 -curvature tensor, W_5 - flat, $\xi - W_5$ flat, $\phi \cdot W_5$ - flat, and W_5 -semi-symmetric

Introduction

In 1989, ^[1] K. Matsumoto introduced Lorentzian paracontact manifolds, specifically LP-Sasakian manifolds, which have since been extensively studied by various geometers. Subsequent research by Matsumoto, Mihai, Rosca, Shaikh, De, Venkatesha, Pradeep Kumar, and Bagewadi focused on these manifolds with significant results ^[3]. In 1995, ^[4] Sinha and Sai Prasad defined para-Kenmotsu and special para-Kenmotsu manifolds, akin to P-Sasakian and SP-Sasakian manifolds. Abdul Haseeb and Rajendra Prasad in 2018 introduced Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifolds, studying ϕ -semisymmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons ^[6]. Njori *et al.* did several studies on W_8 - curvature tensors on various types of manifolds, including Kenmotsu manifolds ^[2, 8].

In 1970, ^[5] Pokhariyal and Mishra introduced new tensor fields on a Riemannian manifold, called the Weyl-projective curvature tensor of type (1, 3) and the tensor field E. The concept of W_5 -curvature tensor was defined by Tripathi and Gupta ^[15] of an n (where $n = 2m + 1$)-dimensional Riemannian manifold are, respectively, defined as

$$W_5(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1} [g(X, Z)S(Y, T) - g(Y, T)S(X, Z)] \quad (2.1)$$

For all $(X, Y, Z) \in_X(M)$. Where R represents the curvature tensor and S corresponds to the Ricci tensor of the manifold.

2. Preliminaries

An n -dimensional differentiable manifold M admitting a (ϕ, ξ, η, g) , (1,1) tensor field ϕ , contravariant vector field ξ , a 1 -form η and the Lorentzian metric g is called Lorentzian almost Paracontact manifold ^[7] if it satisfies:

Corresponding Author:**FN Mburu**

Kirinyaga University, School of
Pure and Applied Sciences,
Kutus, Kirinyaga County,
Kenya

$$\phi^2 X = X + \eta(X)\xi \quad (2.1) \quad \eta(\xi) = -1, \quad (2.2)$$

$$\phi\xi = 0 \quad (2.3)$$

$$\text{rank}\phi = n - 1 \quad (2.4)$$

$$\phi(X, Y) = \phi(Y, X) \quad (2.5)$$

Where

$$\phi(X, Y) = g(X, \phi Y) \quad (2.6)$$

Additionally, in the Lorentzian Para Kenmotsu manifold we have

$$g(X, \phi Y) = g(\phi X, Y) \quad (2.7)$$

$$g(X, \xi) = \eta(X) \quad (2.8)$$

$$g(X, \phi^2 Y) = g(\phi X, \phi Y) \quad (2.9)$$

From (2.1) and (2.9) we have

$$\begin{aligned} g(X, \phi^2 Y) &= g(X, Y + \eta(Y)\xi) \\ &= g(X, Y) + \eta(Y)g(X, \xi) \\ g(\phi X, \phi Y) &= g(X, Y) + \eta(Y)\eta(X) \end{aligned} \quad (2.10)$$

A Lorentzian almost Paracontact manifold M is called Lorentzian Para-Kenmotsu (briefly LP-Kenmotsu) manifold (2018) [3] if

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \quad (2.11)$$

From 2.11 replacing Y by ϕY

$$(\nabla_X \phi)\phi Y = -g(\phi X, \phi Y)\xi - \eta(\phi Y)\phi X, \quad (2.12)$$

For any vector fields X , and Y on M and ∇ is the operator of covariant differentiation with respect to the Lorentzian metric g .

Furthermore, on the LP-Kenmotsu manifold, the following relations hold: [4]

$$\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi \quad (2.13)$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y) \quad (2.14)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \quad (2.15)$$

$$R(\xi, X)\xi = X + \eta(X)\xi = -\nabla_X \xi, \quad (2.16)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.17)$$

$$S(X, \xi) = (n - 1)\eta(X), \quad (2.18)$$

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.19)$$

$$Q\xi = (n - 1)\xi \quad (2.20)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \quad (2.21)$$

$$S(X, Y) = g(QX, Y)$$

Where Q is Ricci operator defined as $QX = (n - 1)X$

For any vector fields X, Y, Z on M and where S is Ricci Tensor and Q , Ricci operator and R Curvature tensor with respect to Levi-Civita connection ∇ .

A Lorentzian Para-Kenmotsu manifold M is said to be an η – Einstein manifold if its Ricci-tensor $S(X, Y)$ is of the form (2021) ^[7]

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \quad (2.22)$$

where a and b are scalar functions on M . In particular, if $b = 0$, then the manifold is said to be an Einstein manifold. Additionally, special η – Einstein manifold when $a = 0$.

3.A W_5 -flat L.P Kenmotsu manifold

Definition 3.1, An n -dimensional L. P. Kenmotsu manifold M is said to be flat if it satisfies the following condition:

$$R(X, Y)Z = 0$$

Theorem 3.1, W_5 -flat L.P -Kenmotsu manifold is a flat manifold.

Proof

Suppose the LP-Kenmotsu manifold is W_5 -flat, then the following hold

begin{equation}

$$\therefore W_5(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1}(g(X, Z)S(Y, T) - S(X, Z)g(Y, T)) = 0$$

$$R(X, Y, Z, T) + \frac{1}{n-1}(g(X, Z)S(Y, T) - S(X, Z)g(Y, T)) = 0$$

$$R(X, Y, Z, T) + (g(X, Z)g(Y, T) - g(X, Z)g(Y, T)) = 0$$

$$R(X, Y, Z, T) = 0$$

This completes the proof that a W_5 -LP-Kenmotsu manifold is a flat manifold

4. A ξ – W_5 -flat Lorentzian Para Kenmotsu manifold

Definition 4.1: An n -dimensional LP-Kenmotsu manifold is said to be ξ – W_5 -flat if its W_5 -curvature tensor satisfies the following condition.

$$W_5(X, Y)\xi = 0$$

Theorem 4.1: An ξ - W_5 flat Lorentzian Para Kenmotsu manifold is a special type of η -Einstein manifold.

Let us consider

$$W_5(X, Y)\xi = R(X, Y)\xi + \frac{1}{n-1}\{g(X, \xi)QY - S(X, \xi)Y\} = 0 \quad (4.1)$$

$$0 = \eta(Y)X - \eta(X)Y + g(X, \xi)Y - g(X, \xi)Y \quad (4.2)$$

$$\therefore \eta(X)Y = \eta(Y)X \quad (4.3)$$

$$\eta(X)g(Y, U) = \eta(Y)g(X, U) \quad (4.4)$$

Setting $Y=\xi$

$$\eta(X)g(\xi, U) = \eta(\xi)g(X, U) \quad (4.5)$$

$$\eta(X)\eta(U) = -g(X, U)$$

$$g(X, U) = -\eta(X)\eta(U)$$

$$S(X, U) = -(n-1)\eta(X)\eta(U) \quad (4.6)$$

Hence (4.6) is a special η -Einstein manifold

W_3 – Semisymmetric Lorentzian Para Kenmotsu manifold

Definition 5.1: An n -dimensional W_5 – Lorentzian-Para-Kenmotsu manifold M is called Semisymmetric if for all vector fields X, Y, Z, U, V on M , the following holds:

$$(R(X, Y) \cdot R)(U, V)Z = 0. \quad (5.1)$$

Definition 5.2 An n -dimensional W_5 – Lorentzian-Para-Kenmotsu manifold M is called

W_5 -semisymmetric if, for all vector fields X, Y, Z, U, V on M , the following holds:

$$(R(X, Y) \cdot W_5)(U, V)Z = 0. \quad (5.2)$$

Where $R(X, Y)$ serves as a derivation on the curvature tensor W_5 .

Theorem 5.1: Any Lorentzian Para-Kenmotsu manifold is semisymmetric.

Theorem 5.2 Any W_5 -Lorentzian Para-Kenmotsu manifold is W_5 -semisymmetric.

Proof of Theorems

Proof of Theorem 5.1

Proving that

$$(R(X, Y) \cdot R)(U, V)Z = 0. \quad (5.3)$$

Hence, expanding $(R(X, Y) \cdot R)(U, V)Z$ we get

$$(R(X, Y) \cdot R)(U, V)Z = R(X, Y)R(U, V)Z - R(R(X, Y)U, V)Z - R(U, R(X, Y)V)Z - R(U, V)R(X, Y)Z \quad (5.4)$$

Putting $U = \xi$ in (5.4) above gives

$$(R(X, Y)R) \cdot R(\xi, V)Z = R(X, Y)R(\xi, V)Z - R(R(X, Y)\xi, V)Z - R(\xi, R(X, Y)V)Z - R(\xi, V)R(X, Y)Z \quad (5.5)$$

Simplifying each of the four terms in (5.5) separately yields

First term: $R(X, Y)R(\xi, V)Z$

$$R(X, Y)R(\xi, V)Z = R(X, Y)[g(V, Z)\xi - \eta(Z)V]$$

$$R(X, Y)R(\xi, V)Z = [g(Y, g(V, Z)\xi)X - g(X, g(V, Z)\xi)Y - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y]$$

$$R(X, Y)R(\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5.6)$$

Second Term: $R(R(X, Y)\xi, V)Z$

$$R(R(X, Y)\xi, V)Z = R([\eta(Y)X - \eta(X)Y], V)Z$$

$$= g(V, Z)\eta(Y)X - g(V, Z)\eta(X)Y - g([\eta(Y)X - \eta(X)Y], Z)V$$

$$\therefore R(R(X, Y)\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Y)g(X, Z)V + \eta(X)g(Y, Z)V \quad (5.7)$$

Third term: $R(\xi, R(X, Y)V)Z$

$$R(\xi, R(X, Y)V)Z = R(\xi, [g(Y, V)X - g(X, V)Y])Z$$

$$= g([\eta(Y)X - \eta(X)Y], Z)\xi - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y$$

$$\therefore R(\xi, R(X, Y)V)Z = g(X, Z)g(Y, V)\xi - g(Y, Z)g(X, V)\xi - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5.8)$$

Fourth term: $R(\xi, V)R(X, Y)Z$

$$R(\xi, V)R(X, Y)Z = g(V, R(X, Y)Z)\xi - \eta(R(X, Y)Z)V$$

$$= g(V, [g(Y, Z)X - g(X, Z)Y])\xi - \eta([g(Y, Z)X - g(X, Z)Y])V$$

$$\therefore R(\xi, V)R(X, Y)Z = g(V, X)g(Y, Z)\xi - g(V, Y)g(X, Z)\xi - \eta(X)g(Y, Z)V + \eta(Y)g(X, Z)V \quad (5.9)$$

Putting equations (5.6), (5.7), (5.8) and (5.9) in equation 5.5 gives

$$R(X, Y)R(\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5.6)$$

$$\therefore R(R(X, Y)\xi, V)Z = \eta(Y)g(V, Z)X - \eta(X)g(V, Z)Y - \eta(Y)g(X, Z)V + \eta(X)g(Y, Z)V \quad (5.7)$$

$$\therefore R(\xi, R(X, Y)V)Z = g(X, Z)g(Y, V)\xi - g(Y, Z)g(X, V)\xi - \eta(Z)g(Y, V)X + \eta(Z)g(X, V)Y \quad (5.8)$$

$$\therefore R(\xi, V)R(X, Y)Z = g(V, X)g(Y, Z)\xi - g(V, Y)g(X, Z)\xi - \eta(X)g(Y, Z)V + \eta(Y)g(X, Z)V \quad (5.9)$$

$$\therefore (R(X, Y) \cdot R)(\xi, V)Z = 0 \quad (5.10)$$

This completes the proof that Lorentzian Para-Kenmotsu is a semisymmetric manifold.

Proof Theorem 5.2

Consider

$$R(X, Y)W_5(U, W)Z = 0 \quad (5.11)$$

$$W_5(X, Y)\xi = \eta(Y)X - \eta(X)Y + \frac{1}{n-1}(g(X, \xi)Y - S(X, \xi)Y) \quad (5.12)$$

Setting $X = \xi$

$$W_5(\xi, Y)\xi = \eta(Y)\xi - \eta(\xi)Y + \frac{1}{n-1}(g(\xi, \xi)Y - S(\xi, \xi)Y) \quad (5.13)$$

$$W_5(\xi, Y)\xi = \eta(Y)\xi - \eta(\xi)Y$$

For semisymmetric,

$$R(X, Y)W_5(U, W)Z = R(X, Y)W_5(U, W)Z - W_5R(X, Y)U, W)Z - W_5(U, R(X, Y)W)Z - W_5(U, W)R(X, Y)Z = 0 \quad (5.14)$$

For any $X, Y, U, W, Z \in x(M)$

$$\text{Taking } X = Z = \xi \text{ and applying} \quad (5.14)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad (5.15)$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \quad (5.16)$$

Expanding each term in (5.14)

First term $R(X, Y)W_5(U, W)Z$

$$R(X, Y)W_5(U, W)Z = g(Y, W_5(U, W)Z)X - g(X, W_5(U, W)Z)Y \quad (5.17)$$

But

$$W_5(U, W)Z = g(W, Z)U - g(U, Z)W$$

$$W_5(U, W)Z = g(Y, g(W, Z)U - g(U, Z)W)X - g(X, g(W, Z)U - g(U, Z)W)Y \quad (5.18)$$

$$R(X, Y)W_5(U, W)Z = g(Y, U)g(W, Z)X - g(Y, W)g(U, Z)X - g(X, U)g(W, Z)Y - g(X, W)g(U, Z)Y \quad (5.19)$$

Second term $W_5R(X, Y)U, W)Z$

$$R(X, Y)U = t = g(Y, U)X - g(X, U)Y$$

$$W_5R(X, Y)U, W)Z = W_5(t, W)Z = g(W, Z)t - g(t, Z)W$$

$$W_5R(X, Y)U, W)Z = g(W, Z)[g(Y, U)X - g(X, U)Y] - g([g(Y, U)X - g(X, U)Y], Z)W$$

$$gW_5R(X, Y)U, W)Z = (W, Z)g(Y, U)X - g(W, Z)g(X, U)Y - g(X, Z)g(Y, U)W + g(Y, Z)g(X, U)W \quad (5.20)$$

Third term $W_5(U, R(X, Y)W)Z$

$$W_5(U, R(X, Y)W)Z = W_5(U, m)Z = g(m, Z)U - g(U, Z)m$$

$$R(X, Y)W = g(Y, W)X - g(X, W)Y$$

$$g([g(Y, W)X - g(X, W)Y], Z)U - g(U, Z)[g(Y, W)X - g(X, W)Y]$$

$$W_5(U, R(X, Y)W)Z = g(X, Z)[g(Y, W)U - g(Y, Z)g(X, W)U - g(U, Z)g(Y, W)X + g(U, Z)g(X, V)Y] \quad (5.21)$$

Forth term $W_5(U, W)R(X, Y)Z$

$$R(X, Y)Z = f = g(Y, Z)X - g(X, Z)Y$$

$$W_5(U, W)R(X, Y)Z = g(W, f)U - g(U, f)W$$

$$W_5(U, W)R(X, Y)Z = g(W, [g(Y, Z)X - g(X, Z)Y])U - g(U, [g(Y, Z)X - g(X, Z)Y])W$$

$$W_5(U, W)R(X, Y)Z = g(W, X)g(Y, Z)U - g(W, Y)g(X, Z)U - g(U, X)g(Y, Z)W - g(U, Y)g(X, Z)W \quad (5.22)$$

Combining (5.19)(5.20),(5.21) and (5.22) additionally, set $U = \xi$

$$R(X, Y)W_5(\xi, W)Z - W_5R(X, Y)\xi, W)Z - W_5(\xi, R(X, Y)W)Z - W_5(\xi, W)R(X, Y)Z$$

$$= [g(Y, \xi)g(W, Z) - g(Y, W)g(\xi, Z)X - g(X, \xi)g(W, Z)Y - g(X, W)g(\xi, Z)Y] - [g(W, Z)g(Y, \xi)X - g(W, Z)g(X, \xi)Y - g(X, Z)g(Y, \xi)W \\ + g(Y, Z)g(X, \xi)W] - [g(X, Z)[g(Y, W)U\xi - g(Y, Z)g(X, W)\xi - g(\xi, Z)g(Y, W)X + g(\xi, Z)g(X, V)Y] - [g(W, X)g(Y, Z)\xi - g(W, Y)g(X, Z)\xi \\ - g(U, X)g(Y, Z)W + g(\xi, Y)g(X, Z)W] = 0$$

Hence theorem proved.

6. W_5 -Para Kenmotsu Manifold satisfying the condition $W_5 \cdot R = 0$

Definition 6.1 An n-dimensional W_5 - Lorentzian-Para-Kenmotsu manifold M is said to satisfy $W_5 \cdot R = 0$ condition if

$$(W_5(U, V) \cdot R)(X, Y)Z = 0 \quad (6.1)$$

For any vector fields X, Y, Z, U, V on M .

Theorem 6.1, A W_5 -Lorentzian-para-Kenmotsu manifold satisfies the condition $W_5 \cdot R = 0$

Corollary 6.2: $W_5 \cdot R$ - Lorentzian-para-Kenmotsu manifold is a W_5 - semisymmetric manifold as well as a semisymmetric manifold (where the W_5 -curvature tensor acts as a derivation on the Riemann curvature tensor R)

Proof of theorem 6.1

Proving that the relation (6.1) holds for a W_5 -Lorentzian para-Kenmotsu Manifold

The above equation (6.1) can be written as follows:

$$(W_5(U, V) \cdot R)(X, Y)Z = W_5(U, V)R(X, Y)Z - R(W_5(U, V)X, Y)Z - R(X, W_5(U, V)Y)Z \\ - R(X, Y)W_5(U, V)Z \quad (6.2)$$

Putting $U = \xi$ in (6.2) gives

$$(W_5(\xi, V) \cdot R)(X, Y)Z = W_5(\xi, V)R(X, Y)Z - R(W_5(\xi, V)X, Y)Z - R(X, W_5(\xi, V)Y)Z - R(X, Y)W_5(\xi, V)Z \quad (6.3)$$

But

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y$$

$$W_5(\xi, V)W = g(V, W)\xi - g(\xi, W)V$$

$$W_5(\xi, V)W = g(V, W)\xi - \eta(W)V \quad (6.4)$$

Computing the four terms in (6.3) separately gives

First term: $W_5(\xi, V)R(X, Y)Z$

From (6.4), gives

$$W_5(\xi, V)W = [g(V, [g(Y, Z)X - g(X, Z)Y])\xi - \eta([g(Y, Z)X - g(X, Z)Y])V]$$

$$W_5(\xi, V)R(X, Y)Z = g(V, X)g(Y, Z)\xi - g(V, Y)g(X, Z)\xi - \eta(X)g(Y, Z)V - \eta(Y)g(X, Z)V \quad (6.5)$$

Second Term: $R(W_5(\xi, V)X, Y)Z$

$$W_5(\xi, V)X = g(V, X)\xi - g(\xi, X)V = t$$

$$R(t, Y)Z = g(Y, Z)t - g(t, Z)Y$$

$$R([g(V, X)\xi - g(\xi, X)V], Y)Z = g(Y, Z)[g(V, X)\xi - g(\xi, X)V] - g([g(V, X)\xi - g(\xi, X)V], Z)Y$$

$$R([g(V, X)\xi - g(\xi, X)V], Y)Z = g(Y, Z)g(V, X)\xi - g(Y, Z)g(\xi, X)V - g(\xi, Z)g(V, X)Y + g(V, Z)g(\xi, X)Y$$

$$W_5(\xi, V)R(X, Y)Z = g(Y, Z)g(V, X)\xi - \eta(X)g(Y, Z)V - \eta(Z)g(V, X)Y + \eta(X)g(V, Z)Y \quad (6.6)$$

Third Term: $R(X, W_5(\xi, V)Y)Z$

$$R(X, W_5(\xi, V)Y)Z = g(W_5(\xi, V)Y, Z)X - g(X, Z)W_5(\xi, V)Y$$

$$R(X, W_5(\xi, V)Y)Z = g([g(V, Y)\xi - \eta(Y)V], Z)X - g(X, Z)[g(V, Y)\xi - \eta(Y)V]$$

$$R(X, W_5(\xi, V)Y)Z = \eta(Z)g(V, Y)X - \eta(Y)g(V, Z)X - g(X, Z)g(V, Y)\xi + \eta(Y)g(X, Z)V \quad (6.7)$$

Fourth Term: $R(X, Y)W_5(\xi, V)Z$

$$W_5(\xi, V)Z = g(V, Z)\xi - \eta(Z)V$$

$$R(X, Y)W_5(\xi, V)Z = g(Y, [g(V, Z)\xi - \eta(Z)V])X - g(X, [g(V, Z)\xi - \eta(Z)V])Y$$

$$R(X, Y)W_5(\xi, V)Z = \eta(Y)g(V, Z)X - \eta(Z)g(Y, V)X - \eta(X)g(V, Z)Y + \eta(Z)g(X, V)Y \quad (6.8)$$

Plugging in equations (6.5), (6.6), (6.7) and (6.8) in (6.3) gives

$$\therefore W_5 \cdot R = 0 \quad (6.9)$$

This completes the proof of the theorem

(6.9)

7. A $\phi - W_5$ -LP-Kenmotsu manifold

Definition 7.1: An n-dimensional LP-Kenmotsu manifold is said to be $\phi - W_5$ -flat if W_5 -curvature tensor satisfies the following condition

$$W_5(X, Y) \cdot \phi Z = W_5(X, Y)\phi Z - \phi(W_5(X, Y)Z) = 0 \quad (7.1)$$

Theorem 7: A W_5 -LP-Kenmotsu -manifold is a $\phi - W_5$ flat manifold

Proof.

Consider $\phi - W_5$ LP-Kenmotsu manifold, then the following hold

$$\begin{aligned} & W_5(X, Y)\phi Z - \phi(W_5(X, Y)Z) \\ &= R(X, Y)\phi Z - g(Y, Z)\phi X + g(X, Z)\phi Y + g(X, Y)\phi Z - g(Y, \phi Z)X - g(X, Y)\phi Z + g(Y, Z)\phi X \\ &= R(X, Y)\phi Z + g(X, Z)\phi Y - g(Y, \phi Z)X \end{aligned} \quad (7.2)$$

$$R(X, Y)\phi Z = g(Y, \phi Z)X - g(X, Z)\phi Y \quad (7.3)$$

$$= g(Y, \phi Z)X - g(X, \phi Z)Y - g(Y, Z)\phi X + g(X, Z)\phi Y + g(X, Y)\phi Z - g(Y, \phi Z)X - g(X, Y)\phi Z + g(Y, Z)\phi X \quad (7.4)$$

$$= -g(X, \phi Z)Y + g(X, Z)\phi Y \quad (7.5)$$

Putting $Y = Z = \xi$ in above equation yields

$$W_5(X, \xi) \cdot \phi \xi = 0$$

This completes the proof.

8. A $W_5 - Q$ L.P-Kenmotsu manifold

Definition; W_5 LP Kenmotsu manifold is such that

$$W_5 \cdot Q = 0$$

Theorem 8.1; An n-dimension L.P Kenmotsu manifold satisfy the condition

$$W_5.Q = 0$$

Proof;

Consider $W_5.Q = 0$

Therefore

$$W_5(X, Y)QZ = W_5(X, Y)QZ - Q(W_5)(X, Y)Z \quad (8.1)$$

$$= R(X, Y)QZ - Q(R(X, Y)Z) - \frac{1}{n-1} [S(X, Z)QY - g(X, QZ)QY] + \frac{1}{n-1} [S(X, Z)QY - Q(g(X, Z)QY)] \quad (8.2)$$

$$= R(X, Y)QZ - Q(R(X, Y)Z) - \frac{1}{n-1} (S(X, Z)QY - S(X, Z)QY - (g(X, QZ)QY + Q(g(X, Z)QY)) \quad (8.3)$$

$$R(X, Y)QZ - Q(R(X, Y)Z) - \frac{1}{n-1} [Q(g(X, Z)QY - g(X, QZ)QY] \quad (8.4)$$

But

$$R(X, Y)QZ = g(Y, QZ)X - g(X, QZ)Y$$

$$[g(Y, QZ)X - g(X, QZ)Y] - [g(Y, Z)QX - g(X, Z)QY] - \frac{1}{n-1} [Q(g(X, Z)QY - g(X, QZ)QY] \quad (8.5)$$

$$= [g(Y, QZ)X - g(X, QZ)Y] - [g(Y, Z)QX - g(X, Z)QY] - [Q(g(X, Z)QY - g(X, QZ)QY] \quad (8.6)$$

$$= g(Y, QZ)X - g(Y, Z)QX$$

$$= (n-1)g(Y, Z)X - (n-1)g(Y, Z)X$$

$$= 0$$

9. An n-dimensional L.P Kenmotsu manifold satisfying $Q - W_5 = 0$

Definition 9.1; An n-dimensional L. P Kenmotsu manifold is said to satisfy the condition

$$Q - W_5 = 0 \text{ if } Q.W_5(X, Y)Z = 0.$$

Theorem 9.1; A Lorentzian para kenmotsu manifold satisfying the condition $Q.W_5 = 0$ is a special type of η -Einstein manifold.

Considering the Lorentzian Para Kenmotsu manifold which satisfies the condition

$$Q.W_5(X, Y)Z = 0$$

Therefore

$$Q.W_5(X, Y)Z - W_5(QX, Y)Z - W_5(X, QY)Z - W_5(X, Y)QZ = 0 \quad (9.1)$$

Computing the four terms separately

First term

$$Q(W_5(X, Y)Z) = g(Y, Z)QX - g(X, Z)QY - \frac{1}{n-1} [S(X, Z)QY - Qg(X, Z)QY] \quad (9.2)$$

$$= g(Y, Z)QX - g(X, Z)QY - g(X, Z)QY + g(X, Z)QY \quad (9.3)$$

$$Q(W_5(X, Y)Z) = g(Y, Z)QX - g(X, Z)QY \quad (9.4)$$

Second term

$$W_5(QX, Y)Z = g(Y, Z)QX - g(QX, Z)Y - g(QX, Z)Y \quad (9.5)$$

$$= g(Y, Z)QX - g(QX, Z)Y - g(QX, Z)Y + g(X, Z)QY - g(X, Z)QY \quad (9.6)$$

$$W_5(QX, Y)Z = g(Y, Z)QX - 2g(QX, Z)Y + g(X, Z)QY \quad (9.7)$$

Third term

$$W_5(X, QY)Z = g(QY, Z)X - g(X, Z)QY - [g(X, QZ)Y - g(QX, Z)Y] \quad (9.8)$$

$$= g(Y, QZ)X - g(X, Z)QY \quad (9.9)$$

Fourth term

$$W_5(X, Y)QZ$$

$$W_5(X, Y)QZ = g(Y, QZ)X - g(X, QZ)Y - [g(X, Z)QY - g(X, QZ)Y] \quad (9.10)$$

$$= g(Y, QZ)X - g(X, Z)QY \quad (9.11)$$

Combining the four terms

$$g(Y, Z)QX - g(X, Z)QY - g(Y, Z)QX - 2g(QX, Z)Y + g(X, Z)QY - [g(Y, QZ)X - g(X, Z)QY] - [g(Y, QZ)X - g(X, Z)QY] \quad (9.12)$$

$$= 2g(QX, Z)Y - g(Y, QZ)X - g(Y, QZ)X$$

$$= 2[g(QX, Z)Y - g(Y, QZ)X]$$

$$= 2[g(QX, Z)Y - g(Y, QZ)X] = 0 \quad (9.13)$$

Let $Y = \xi$ and taking inner product of ξ gives

$$2[g(X, Z)\xi - g(\xi, Z)X] = 0 \quad (9.14)$$

But

$$S(X, \xi)Y = (n-1)g(X, \xi)Y \quad (9.15)$$

And

$$g(X, Z)\xi = \eta(X)\eta(Z) \quad (9.16)$$

Therefore

$$g(X, \xi)Y = \frac{S(X, \xi)Y}{n-1} \quad (9.17)$$

$$\frac{S(X, \xi)Y}{n-1} = \eta(X)\eta(Y) \quad (9.18)$$

$$S(X, \xi)Y = (n-1)\eta(X)\eta(Y) \quad (9.19)$$

10.0. An n-dimensional LP-Kenmotsu manifold satisfying the condition $W_5 \cdot W_5 = 0$

Definition 10.1 An n-dimensional LP-Kenmotsu manifold is said to satisfy the condition $W_5 \cdot W_5 = 0$ if

$$W_5 \cdot W_5 = W_5(U, V) \cdot W_5(X, Y)Z = 0$$

Theorem 10.1: An LP-Kenmotsu manifold satisfying the condition $W_5 \cdot W_5 = 0$ is a flat manifold

Proof:

$$\text{Consider } W_5 \cdot W_5 = W_5(U, V) \cdot W_5(X, Y)Z = 0$$

Where,

$$W_5(U, V) \cdot W_5(X, Y)Z = W_5(U, V)W_5(X, Y)Z - W_5(W_5(U, V)X, Y)Z - W_5(X, W_5(U, V)Y)Z$$

$$-W_5(X, Y)W_5(U, V)Z = 0 \quad (10.1)$$

Putting $V = \xi$ in (10.1) above we get

$$W_5(U, \xi)W_5(X, Y)Z - W_5(W_5(U, \xi)X, Y)Z - W_5(X, W_5(U, \xi)Y)Z$$

$$-W_5(X, Y)W_5(U, \xi)Z = 0 \quad (10.2)$$

Simplifying each term in (10.2) separately gives

Term 1: $W_5(U, \xi)W_5(X, Y)Z$

$$\text{Let } W_5(X, Y)Z = W = g(Y, Z)X - g(X, Z)Y$$

$$R(U, \xi)W = g(\xi, W)U - g(U, W)\xi$$

$$W_5(U, \xi)W = g(\xi, W)U - g(U, W)\xi + \frac{1}{n-1} [g(U, W)Q\xi - S(U, W)\xi]$$

$$W_5(U, \xi)W = g(\xi, W)U - g(U, W)\xi$$

$$W_5(U, \xi)W = g(\xi, g(Y, Z)X - g(X, Z)Y), U - g(U, [g(Y, Z)X - g(X, Z)Y])\xi$$

$$W_5(U, \xi)W = g(\xi, X)g(Y, Z)U - g(\xi, Y)g(X, Z)U - g(U, X)g(Y, Z)\xi + g(U, Y)g(X, Z)\xi \quad W_5(U, \xi)W = \eta(X)g(Y, Z)U - \eta(Y)g(X, Z)U - g(U, X)g(Y, Z)\xi + g(U, Y)g(X, Z)\xi \quad (10.3)$$

Term II: $W_5(W_5(U, \xi)X, Y)Z$

Applying the definition of W_5 on above, we get

$$W_5(W_5(U, \xi)X, Y)Z = g(Y, Z)W_5(U, \xi)X - g(W_5(U, \xi)X, Z)Y + \frac{1}{n-1} [g(W_5(U, \xi)X, Z)QY - S(W_5(U, \xi)X, Z)Y]$$

$$W_5(W_5(U, \xi)X, Y)Z = W_5(f, Y)Z = g(Y, Z)f - g(f, Z)Y$$

$$W_5(U, \xi)X = f = g(\xi, X)U - g(U, X)\xi$$

$$W_5(W_5(U, \xi)X, Y)Z = g(Y, Z)[g(\xi, X)U - g(U, X)\xi] - g([g(\xi, X)U - g(U, X)\xi], Z)Y$$

$$W_5(W_5(U, \xi)X, Y)Z = g(Y, Z)g(\xi, X)U - g(Y, Z)g(U, X)\xi - g(U, Z)[g(\xi, X)Y - g(\xi, Z)g(U, X)Y]$$

$$W_5(W_5(U, \xi)X, Y)Z = g(Y, Z)\eta(X)U - g(Y, Z)g(U, X)\xi - g(U, Z)\eta(X)Y + \eta(Z)g(U, X)Y \quad (10.4)$$

Term III: $W_5(X, W_5(U, \xi)Y)Z$

$$W_5(X, W_5(U, \xi)Y)Z = R(X, W_5(U, \xi)Y)Z$$

$$W_5(U, \xi)Y = t = g(\xi, Y)U - g(U, Y)\xi$$

$$R(X, W_5(U, \xi)Y)Z = R(X, t)Z = g(t, Z)X - g(X, Z)t$$

$$R(X, W_5(U, \xi)Y)Z = R(X, [g(\xi, Y)U - g(U, Y)\xi])Z = g([g(\xi, Y)U - g(U, Y)\xi], Z)X - g(X, Z)[g(\xi, Y)U - g(U, Y)\xi]$$

$$R(X, W_5(U, \xi)Y)Z = g(U, Z)(g(\xi, Y)X - g(\xi, Z)g(U, Y))X - g(X, Z)g(\xi, Y)U + g(X, Z)g(U, Y)\xi$$

$$R(X, W_5(U, \xi)Y)Z = g(U, Z)\eta(Y)X - \eta(Z)g(U, Y)X - g(X, Z)\eta(Y)U + g(X, Z)g(U, Y)\xi \quad (10.5)$$

Term IV: $W_5(X, Y)W_5(U, \xi)Z$

$$W_5(X, Y)W_5(U, \xi)Z = R(X, Y)W_5(U, \xi)Z$$

$$W_5(U, \xi)Z = B = g(\xi, Z)U - g(U, Z)\xi$$

$$W_5(X, Y)B = g(Y, B)X - g(X, B)Y$$

$$W_5(X, Y)[g(\xi, Z)U - g(U, Z)\xi] = g(Y, [g(\xi, Z)U - g(U, Z)\xi])X - g(X, [g(\xi, Z)U - g(U, Z)\xi])Y$$

$$W_5(X, Y)W_5(U, \xi)Z = g(Y, U)g(\xi, Z)X - g(Y, \xi)g(U, Z)X - g(X, U)g(\xi, Z)Y - g(X, \xi)g(U, Z)Y$$

$$W_5(X, Y)W_5(U, \xi)Z = g(Y, U)\eta(Z)X - \eta(Y)g(U, Z)X - g(X, U)\eta(Z)Y + \eta(X)g(U, Z)Y \quad (10.6)$$

Combing the four terms

First term $W_5(U, \xi)W = \eta(X)g(Y, Z)U - \eta(Y)g(X, Z)U - g(U, X)g(Y, Z)\xi + g(U, Y)g(X, Z)\xi$

Second term $W_5(W_5(U, \xi)X, Y)Z = g(Y, Z)\eta(X)U - g(Y, Z)g(U, X)\xi - g(U, Z)\eta(X)Y + \eta(Z)g(U, X)Y$

Third term $R(X, W_5(U, \xi)Y)Z = g(U, Z)\eta(Y)X - \eta(Z)g(U, Y)X - g(X, Z)\eta(Y)U + g(X, Z)g(U, Y)\xi$

Fourth term $W_5(X, Y)W_5(U, \xi)Z = g(Y, U)\eta(Z)X - \eta(Y)g(U, Z)X - g(X, U)\eta(Z)Y + \eta(X)g(U, Z)Y$

Therefore,

$$W_5(U, \xi)W - W_5(W_5(U, \xi)X, Y)Z - R(X, W_5(U, \xi)Y)Z - W_5(X, Y)W_5(U, \xi)Z = 0$$

This completes proof of the theorem that an LP-Kenmotsu manifold satisfying $W_5 \cdot W_5 = 0$ is a flat manifold

References

1. Matsumoto K, Mihai I. On a certain transformation in a Lorentzian para Sasakian manifold. Tensor (N.S.). 1988;47(2):189-97.
2. Mishra RS. On Sasakian manifold. Indian J Pure Appl Math. 1972;3(5):739-749.
3. Pokhariyal GP. On symmetric Sasakian manifold. Kenya J Sci Ser A. 1988;9(1-2):39-42.
4. Pokhariyal GP, Mishra RS. Curvature tensor and their relativistic significance II. Yokohama Math J. 1971;19(2):97-103.
5. De UC, Guha N. Conharmonic recurrent Sasakian manifold. Indian J; 1992, p. 209-15.
6. Tripathi MM, De UC. Lorentzian almost para-contact manifolds and their submanifolds. Pure Appl Math. 2001;8(2):101-125.
7. Njori PW, Moindi SK, Pokhariyal GP. A study on W_6 -curvature tensors and W_8 -curvature tensors in Kenmotsu manifolds admitting semi-symmetric metric connection. 2021;(1):41-57.
8. Njori PW. Curvature tensors on semi-Riemannian and generalized Sasakian space forms admitting semi-symmetric metric connection [dissertation]. Nairobi: University of Nairobi; 2018.
9. Satyanarayana T, Prasad KS. On semi-symmetric para Kenmotsu manifolds. Turk J Anal Number Theory. 2015;3(6):145-148.
10. Pokhariyal GP. Relativistic significance of curvature tensors. Int J Math Math Sci. 1982;5:133-139.
11. Prasad RAJENDRA, Haseeb ABDUL, Gautam UK. On ϕ -semisymmetric LP-Kenmotsu manifolds with a qsnm-connection admitting Ricci solitons. Kragujevac J Math. 2021;45(5):815-827.
12. Njori P, Mutinda F. Study of some new curvature tensors on Lorentzian para Sasakian manifolds and other related manifolds; 2021.
13. Yoldas H, Yasar E. Some notes on Kenmotsu manifold. Facta Univ Ser Math Inform. 2021;1:949-961.
14. Devi SS, Prasad KS, Satyanarayana T. Certain curvature conditions on Lorentzian para-Kenmotsu manifolds. Reliab Theory Appl. 2022;17(2):413-421.
15. Mert T, Ateken M. On some important characterizations of Lorentz para-Kenmotsu manifolds on some special curvature tensors. Asian-Eur J Math. 2024;17(1):2350246.
16. Bilal M, Raghuwanshi T, Pandey MK, Haseeb A, Goyal A. On generalized Weyl conformal curvature tensor in para-Kenmotsu manifolds: On generalized Weyl conformal curvature tensor. Results Nonlinear Anal. 2024;7(3):55-64.