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On fuzzy context of tri-parameterized generalized entropy measure linked to conditional description

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Abstract

Based on a review of the literature on fuzzy information measures, there are numerous well-known parametric and non-parametric measures, along with their characteristics and uses. The fuzzy entropy metric is often used to characterize and quantify fuzzy uncertainty and ambiguity. We provide a novel parametric exponential entropy for this measure on a fuzzy set. Some properties are also provided after demonstrating the correctness of the suggested entropy measure.

Keywords: Shannon measure, parametric entropy, fuzzy set, exponential entropy

Introduction

Applications of information theory [18, 20, 21, 22] and, more recently, fuzzy set theory [11, 12, 13] have been found in many different domains. In this paper, we assessment, such as fuzzy entropy. Since fuzzy entropy can be approached in a variety of ways, we go over each one independently. An overview of information theory and fuzzy logic is also given.

Probability Entropy

The fuzzy set theory was discovered in 1965 by L. A. Zadeh [14, 15, 26], who defined an entropy for a fuzzy event as the uncertainty associated with that event. This entropy is called a weighted Shannon entropy, in which the membership values are considered as weights. In his work, Zadeh extended the Shannon entropy to be applied as a fuzzy entropy on a fuzzy subset A for the finite set $X = \{x_1, x_2, \dots, x_n\}$ with respect to the probability distribution $P = \{p_1, p_2, \dots, p_n\}$ [24, 25].

$$H = - \sum_{i=1}^n \mu_A(x_i) p_i \log p_i$$

Where p represents A's membership function and p_i is the likelihood that x_i will occur. This scenario exhibits the three categories of uncertainty: ambiguity, randomness, and vagueness, which translates to fuzziness and randomness.

Our Result

This encourages us to establish a fuzzy entropy function with a form that is more like to Shannon's probabilistic entropy. In light of this, we increase our to provide a fuzzy counterpart of Shannon's entropy, use the probability-Membership Density Function [3, 6, 9] analogy. A discrete random variable $X = \{x_1, x_2, \dots, x_n\}$ with a probability mass function $P = p_1, p_2, \dots, p_n$ is examined, where p_i represents the probability ($X = x_i$). Given X, the probabilistic Shannon's entropy is as follows:

$$I_{sh}^P = - \sum_{i=1}^n p_i \log p_i$$

Where an entropy function that is "probabilistic" is indicated by the letter P in the superscript. The prior probability p_i represents previous knowledge.

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concerning the prevalence of x_i . As a result, $\log\left(\frac{1}{p_i}\right)$ [16, 17, 23] provides an uncertainty metric known as the information gain related to x_i . Shannon's probabilistic entropy is calculated by adding together these information gains and weighing them according to their respective probabilities.

Using the Shannon entropy [8, 9, 19] as a foundation, several generalized entropies have been proposed in the literature and have applications in a variety of fields, including computing, information processing, statistics, economics, etc.

Agrawal and Shannon entropy given by

$$I_{A-Sh} = \frac{1}{n-1} \sum_{i=1}^n (1 - \theta_i) e^{1-\theta_i} \quad (1)$$

After analyzing the standard Shannon information entropy, Pal and Pal [8, 9] proposed an information entropy known as the exponential entropy, which is given by

$$I_{PP}^P = \sum_{i=1}^n p_i e^{1-p_i} \quad (2)$$

In accordance with (2), Kvalseth presented and examined the generalized exponential entropy of order- α , which is provided by

$$I_{PP}^P = \frac{\alpha^{-1} \sum_{i=1}^n p_i e^{1-p_i}}{\sum_{i=1}^n p_i} \quad (3)$$

Using a discrete probabilistic framework, first attempt to quantify the uncertain aspects of a fuzzy event. The definition of the entropy of fuzzy set A with regard to (X, P) seems to have been made by Zadeh [20].

$$I_A = \sum_{i=1}^n \mu_A(x_i) P(x_i) \log P(x_i)$$

De Luca and Termini [2, 7, 10] defined fuzzy entropy for a fuzzy set A corresponding to (1) as

$$I_{LT} = -\frac{1}{n} \sum_{i=1}^n \mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i) \log(1 - \mu_A(x_i)))$$

Afterwards, Bhandari and Pal [1, 4, 5] conducted a survey on fuzzy set information measures and provided several novel fuzzy entropy [10] measurements. According to (2), they've recommended taking the following action:

$$I_{BP} = \frac{K}{1-\alpha} \sum_{i=1}^n \log[\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha]$$

Pal and Pal defined exponential fuzzy entropy for a fuzzy set corresponding (3) as

$$I_{PP} = K \sum_{i=1}^n [\mu_A(x_i) e^{1-\mu_A(x_i)} + (1 - \mu_A(x_i)) e^{\mu_A(x_i)}] \quad (4)$$

Some set operation can be defined as follows:

- $A^C = \{x, 1 - \mu_A(x_i), \mu_A(x_i) / x \in X\}$
- $A \cap B = \{x, \min \mu_A(x_i), \mu_B(x_i) / x \in X\}$
- $A \cup B = \{x, \max \mu_A(x_i), \mu_B(x_i) / x \in X\}$

Where, $A = \{(x, \mu_A(x_i) / x \in X)\}$

$B = \{(x, \mu_B(x_i) / x \in X)\} \& A, B \in F(X)$

De Luca and Termini's [2] fuzzy entropy measure quantifies the degree of equivalence between the fuzzy values $\mu_A(x_1), \mu_A(x_2), \mu_A(x_3), \dots, \mu_A(x_n)$, that is, the $H(A)$ has a maximum value of $n \log 2$ when all fuzzy values are equal, that is, when each $\mu_A(x_i) = \frac{1}{2}$. The greater the equality among, the bigger the value of $H(A)$. Consequently, we have $H(F) = n \log 2$, where the most fuzzy distribution is

$$F = \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}$$

It was given some formulas of entropy, based on which, we propose the following fuzzy entropy measures

$$1) \quad E_F^1 = -\frac{1}{\log 2} \sum_{i=1}^n p_i [\alpha_i \log \alpha_i + (1 - \alpha_i) \log(1 - \alpha_i)]$$

$$2) \quad E_F^2 = 2 \left(\sum_{i=1}^n (p_i \min\{\alpha_i, 1 - \alpha_i\})^q \right)^{\frac{1}{q}}, q \geq 1$$

$$3) \quad E_F^3 = \frac{1}{(\sqrt{e}-1)} (\sum_{i=1}^n p_i [\alpha_i e^{1-\alpha_i} + (1-\alpha_i)e^{\alpha_i} - 1])$$

$$4) \quad E_F^4 = 1 - \sum_{i=1}^n p_i |1 - 2\alpha_i|$$

$$5) \quad E_F^5 = \frac{1}{(1-q)\log 2} \sum_{i=1}^n p_i \log[\alpha_i^q + (1-\alpha_i)^q]; q > 0, q \neq 1$$

$$6) \quad E_F^6 = 4^q \sum_{i=1}^n p_i \alpha_i^q (1-\alpha_i)^q, 0 < q < 1$$

Bhandari & Pal introduced two entropy measures on fuzzy set as

$$E_{pp}(A) = \frac{1}{n(\sqrt{e}-1)} \sum_{i=1}^n [\mu_A(x_i)e^{1-\mu_A(x_i)} + (1-\mu_A(x_i))e^{\mu_A(x_i)} - 1]$$

Later on exponential entropy for fuzzy set is also proposed by Verma and Sharma ^[17] as

$$E_{VS}(A) = \frac{1}{n \left[\left(e^{\frac{1}{2}} \right)^\alpha - 1 \right]} \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\alpha} + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\alpha} - 1]$$

New Proposed Exponential Entropy Measure

Using the preceding notions as a guide, we provide exponential fuzzy entropy measures as

$$E(A) = \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^\alpha - \left[e^{\frac{1}{2}} \right]^\beta - \left[e^{\frac{1}{2}} \right]^\gamma + 1 \right\}} \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\alpha} + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\alpha} - 1] - \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\beta} + ((1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\beta} - 1) - \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\gamma} + ((1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\gamma} - 1)], \alpha, \beta, \gamma > 0 \quad (5)$$

Claim: The measurement specify $E(A)$ on fuzzy set A is a valid entropy measure.

E_1 : Sharpness

To Prove $E(A) = 0$ for $\mu_A(x_i) = 0$ or 1

First we suppose that $E(A) = 0$

$$\Rightarrow \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^\alpha - \left[e^{\frac{1}{2}} \right]^\beta - \left[e^{\frac{1}{2}} \right]^\gamma + 1 \right\}} \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\alpha} + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\alpha} - 1] - \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\beta} + ((1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\beta} - 1) - \sum_{i=1}^n [\mu_A(x_i)e^{(\mu_A(x_i))^\gamma} + ((1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\gamma} - 1)] = 0$$

$$\begin{aligned} &\Rightarrow [\mu_A(x_i)e^{(\mu_A(x_i))^\alpha} + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\alpha} - 1] \\ &= [\mu_A(x_i)e^{(\mu_A(x_i))^\beta} + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\beta} - 1] + [\mu_A(x_i)e^\gamma + (1-\mu_A(x_i))e^{[1-\mu_A(x_i)]^\gamma} - 1] \\ &\Rightarrow \mu_A(x_i)e^{(\mu_A(x_i))^\alpha} + e^{[1-\mu_A(x_i)]^\alpha} - \mu_A(x_i)e^{[1-\mu_A(x_i)]^\alpha} - 1 = \end{aligned}$$

$$\mu_A(x_i)e^{(\mu_A(x_i))^\beta} + e^{[1-\mu_A(x_i)]^\beta} - \mu_A(x_i)e^{[1-\mu_A(x_i)]^\beta} + \mu_A(x_i)e^{(\mu_A(x_i))^\gamma} + e^{[1-\mu_A(x_i)]^\gamma} - \mu_A(x_i)e^{[1-\mu_A(x_i)]^\gamma} - 2$$

$$\begin{aligned} &\Rightarrow \mu_A(x_i) [e^{(\mu_A(x_i))^\alpha} - e^{[1-\mu_A(x_i)]^\alpha}] + \\ &e^{[1-\mu_A(x_i)]^\alpha} - 1 = \mu_A(x_i) [e^{(\mu_A(x_i))^\beta} + e^{(\mu_A(x_i))^\gamma} - e^{[1-\mu_A(x_i)]^\beta} - e^{[1-\mu_A(x_i)]^\gamma}] + \\ &e^{[1-\mu_A(x_i)]^\beta} + e^{[1-\mu_A(x_i)]^\gamma} - 2 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \mu_A(x_i) [e^{(\mu_A(x_i))^\alpha} - e^{(\mu_A(x_i))^\beta} - e^{(\mu_A(x_i))^\gamma}] + 1 \\ &= \mu_A(x_i) [e^{[1-\mu_A(x_i)]^\alpha} - e^{[1-\mu_A(x_i)]^\beta} - e^{[1-\mu_A(x_i)]^\gamma}] + \\ &e^{[1-\mu_A(x_i)]^\beta} + e^{[1-\mu_A(x_i)]^\gamma} - e^{[1-\mu_A(x_i)]^\alpha} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \mu_A(x_i) [e^{(\mu_A(x_i))^\alpha} - e^{(\mu_A(x_i))^\beta} - e^{(\mu_A(x_i))^\gamma}] + 1 \\ &= e^{[1-\mu_A(x_i)]^\alpha} (\mu_A(x_i) - 1) - e^{[1-\mu_A(x_i)]^\beta} (\mu_A(x_i) - 1) - e^{[1-\mu_A(x_i)]^\gamma} (\mu_A(x_i) - 1) \end{aligned}$$

$$\Rightarrow \left[e^{(\mu_A(x_i))^\alpha} - e^{(\mu_A(x_i))^\beta} - e^{(\mu_A(x_i))^\gamma} \right] + 1 = (\mu_A(x_i) - 1)\mu_A(x_i) \left[e^{[1-\mu_A(x_i)]^\alpha} - e^{[1-\mu_A(x_i)]^\beta} - e^{[1-\mu_A(x_i)]^\gamma} \right] \quad (6)$$

This is true if $\mu_A(x_i) = 0$ or 1

Conversely, we assume that

If $\mu_A(x_i) = 0$ or 1 , from equation (5)

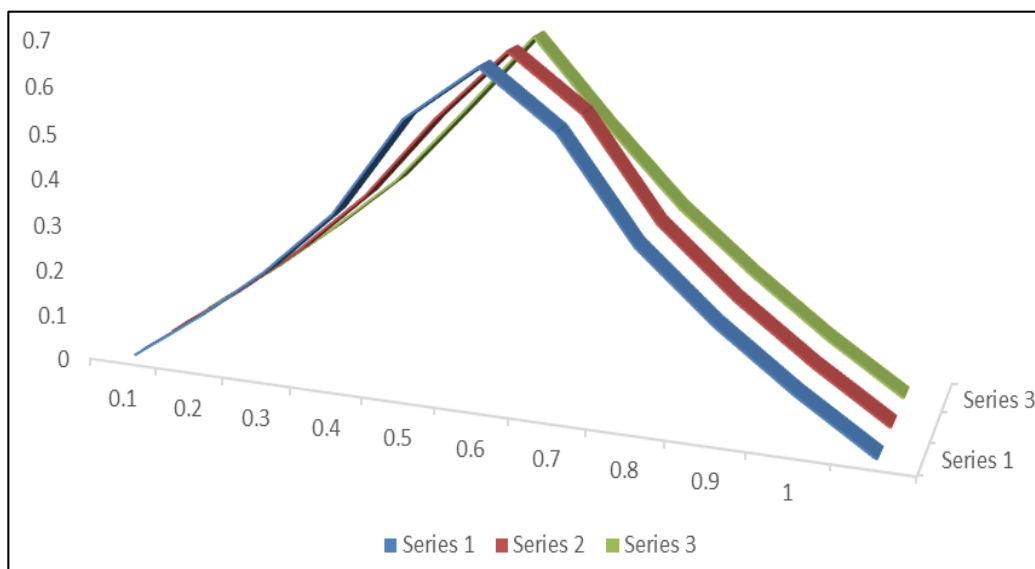
$$\Rightarrow E_A = 0$$

Hence, we can say that

$E(A) = 0$ for $\mu_A(x_i) = 0$ or 1

Next, as illustrated in below Figure, we have graphically depicted the measure (5) with the aid of data. We have fixed, $n = 1$, $\beta = 2$ and $\gamma = 3$ and various value of α for this purpose. Next, we calculated various values of $E(A)$ for various, as indicated in below Table:

$\mu_A(x_i)$	$\alpha=0.0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
$E(A)$ for $\mu_A=2$	0.0000	0.1054	0.2239	0.3648	0.5767	0.6931	0.5767	0.3648	0.2239	0.1054	0.0000
$E(A)$ for $\mu_A=4$	0.0000	0.1053	0.2239	0.3610	0.5397	0.6931	0.5767	0.3640	0.2238	0.1055	0.0000
$E(A)$ for $\mu_A=12$	0.0000	0.1054	0.2238	0.3519	0.5197	0.6931	0.5167	0.3540	0.2235	0.1051	0.0000



(E₂): Maximality

To prove $E(A)$ is Maximum at $\mu_A(x_i) = 0.5$

Now

$$\begin{aligned} \frac{\partial E(A)}{\partial \mu_A(x_i)} &= \frac{1}{n(e^{-2\alpha} - e^{-2\beta} - e^{-2\gamma} + 1)} \sum_{i=1}^n \left\{ \left[e^{(\mu_A(x_i))^\alpha} + \mu_A(x_i)\alpha(e)^{(\mu_A(x_i))^{\alpha-1}} - (e)^{(1-\mu_A(x_i))^{\alpha-1}} + \left[0 - \mu_A(x_i)\alpha(e)^{(1-\mu_A(x_i))^{\alpha-1}} - 0 \right] \right] \right. \\ &\quad - \left[0 - \mu_A(x_i)\gamma(e)^{(1-\mu_A(x_i))^{\gamma-1}} - 0 \right] \\ &\quad - \left[e^{(\mu_A(x_i))^\beta} + \mu_A(x_i)\beta(e)^{(\mu_A(x_i))^{\beta-1}} - (e)^{(1-\mu_A(x_i))^{\beta-1}} \right] \\ &\quad - \left[0 - \mu_A(x_i)\beta(e)^{(1-\mu_A(x_i))^{\beta-1}} - 0 \right] \\ &\quad \left. - \left[e^{(\mu_A(x_i))^\gamma} + \mu_A(x_i)\gamma(e)^{(\mu_A(x_i))^{\gamma-1}} - (e)^{(1-\mu_A(x_i))^{\gamma-1}} \right] \right\} \end{aligned}$$

Case-I: Let $0 \leq \mu_A(x_i) < 0.5$, then we have $\frac{\partial E(A)}{\partial \mu_A(x_i)} > 0 \forall \alpha, \beta > 0$, This gives that $E(A)$ is increasing function.

Case-II: Let $0.5 < \mu_A(x_i) \leq 1$, then we have $\frac{\partial E(A)}{\partial \mu_A(x_i)} < 0 \forall \alpha, \beta > 0$, This gives that $E(A)$ is decreasing function.

Case-III: Let $\mu_A(x_i) = 0.5$, then we have $\frac{\partial E(A)}{\partial \mu_A(x_i)} = 0 \forall \alpha, \beta > 0$

It is evident from above graph that $E(A)$ is a concave function, with a global maximum at $\mu_A(x_i) = 0.5$. Therefore, $E(A)$ is the greatest if the most fuzzy set is A

(E₃): Resolution

To demonstrate that $E(A^*) \leq E(A)$ here A^* is a sharpened form of A . We know from earlier axioms that $E(A)$ is a decreasing function for $0.5 < \mu_A(x_i) \leq 1$ and an increasing function for $0 \leq \mu_A(x_i) < 0.5$, Using the idea above.

$$\mu_{A^*}(x_i) \leq \mu_A(x_i) \Rightarrow E(A^*) \leq E(A) \quad (7)$$

$$\mu_A(x_i) \leq \mu_{A^*}(x_i) \Rightarrow E(A) \leq E(A^*) \quad (8)$$

From (7) and (8)

$\Rightarrow E(A^*) \leq E(A)$ in both cases.

(E₄): Symmetry

Assuming that A^c is A 's complement, we may use the membership function to take $\mu_{A^c}(x_i) = 1 - \mu_A(x_i)$, Then obviously

$$\Rightarrow E(A^c) \leq E(A)$$

This demonstrates the validity of $E(A)$ be as an entropy measure on a fuzzy set.

Few characteristics of entropy measures

Assume that $FS(X)$ represents the family of all fuzzy sets of universe X , where FS represents fuzzy sets and $\zeta, \xi, \chi \in FS(X)$ is given by

- $\zeta = \left[\left(x, \mu_\zeta(x) \right) / x \in X \right]$
- $\xi = \left[\left(x, \mu_\xi(x) \right) / x \in X \right]$
- $\chi = \left[\left(x, \mu_\chi(x) \right) / x \in X \right]$

Then some set operations can be defined as follows:

- a) $\zeta \cup \xi = \left[\left(x, \max(\mu_\zeta(x), \mu_\xi(x)) \right) / x \in X \right]$
- b) $\zeta \cap \xi = \left[\left(x, \min(\mu_\zeta(x), \mu_\xi(x)) \right) / x \in X \right]$
- c) $(\zeta \cup \xi) \cup \chi = \left[\left(x, \max\{\max(\mu_\zeta(x), \mu_\xi(x)), \mu_\chi(x)\} \right) / x \in X \right]$
- d) $(\zeta \cap \xi) \cap \chi = \left[\left(x, \min\{\min(\mu_\zeta(x), \mu_\xi(x)), \mu_\chi(x)\} \right) / x \in X \right]$
- e) $\chi^c = \left[\left(x, \mu_{\chi^c}(x) = 1 - \mu_\chi(x) \right) / x \in X \right]$

The following are some significant characteristics of the exponential entropy measure:

Claim

For Fuzzy set $\zeta, \xi, \chi \in FS(X)$ prove that

$$E((\zeta \cup \xi) \cup \chi) = E(\zeta \cup (\xi \cup \chi))$$

$$E((\zeta \cap \xi) \cap \chi) = E(\zeta \cap (\xi \cap \chi))$$

Proof: We suppose that

$$X_\eta = \{x_i \in X / \mu_\zeta(x_i) \geq \mu_\xi(x_i) \geq \mu_\chi(x_i)\}$$

$$X_\theta = \{x_i \in X / \mu_\zeta(x_i) < \mu_\xi(x_i) < \mu_\chi(x_i)\}$$

Where $\mu_\zeta(x), \mu_\xi(x), \mu_\chi$ be the fuzzy membership function of ζ, ξ, χ respectively.

We know that

$$(i) \quad E(\zeta \cup \xi) = \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^\alpha - \left[e^{\frac{1}{2}} \right]^\beta \right\}} \sum_{x_i \in X} \left[\mu_{\zeta \cup \xi}(x_i) e^{\left(\mu_{\zeta \cup \xi}(x_i) \right)^\alpha} + \left(1 - \mu_{\zeta \cup \xi}(x_i) \right) e^{\left[1 - \mu_{\zeta \cup \xi}(x_i) \right]^\alpha} - 1 \right] -$$

$$\begin{aligned}
& \sum_{X_{\theta}=X} \left[\mu_{\zeta \cup \xi}(x_i) e^{(\mu_{\zeta \cup \xi}(x_i))^{\beta}} + \left((1 - \mu_{\zeta \cup \xi}(x_i)) e^{[1 - \mu_{\zeta \cup \xi}(x_i)]^{\beta}} - 1 \right) \right] \\
&= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} \right\}} \sum_{X_{\eta}=X} \left\{ \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\alpha}} - (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\alpha}} - \right. \\
& \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\beta}} - (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\beta}} \Big\} + \\
& \sum_{X_{\theta}=X} \left\{ \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\alpha}} + (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\alpha}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} - \right. \\
& \left. (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} \right\} \tag{9}
\end{aligned}$$

And

$$\begin{aligned}
E(\zeta \cap \xi) &= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} \right\}} \sum_{X_{\eta}=X} \left\{ \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\alpha}} + (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\alpha}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} - (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} \right\} \\
&+ \sum_{X_{\theta}=X} \left\{ \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\alpha}} + (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\alpha}} - \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\beta}} - (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\beta}} \right\} \tag{10}
\end{aligned}$$

Now, From equation (9), we have

$$\begin{aligned}
E((\zeta \cup \xi) \cup \chi) &= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} - 1 \right\}} \sum_{X_{\eta}=X} \left[\mu_{\zeta \cup \xi}(x_i) e^{(\mu_{\zeta \cup \xi}(x_i))^{\alpha}} + (1 - \mu_{\zeta \cup \xi}(x_i)) e^{[1 - \mu_{\zeta \cup \xi}(x_i)]^{\alpha}} - 1 \right] - \\
& \sum_{X_{\theta}=X} \left[\mu_{\zeta \cup \xi}(x_i) e^{(\mu_{\zeta \cup \xi}(x_i))^{\beta}} + \left((1 - \mu_{\zeta \cup \xi}(x_i)) e^{[1 - \mu_{\zeta \cup \xi}(x_i)]^{\beta}} - 1 \right) + \sum_{X_{\theta}=X} \left[\mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\gamma}} + (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\gamma}} - 1 \right] \right] \\
&= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} - 1 \right\}} \sum_{X_{\eta}=X} \left\{ \left\{ \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\alpha}} + (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\alpha}} - \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\beta}} - (1 - \right. \right. \\
& \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\beta}} \Big\} + \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\alpha}} + (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\alpha}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} - (1 - \\
& \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} \Big\} + \sum_{X_{\theta}=X} \left[\mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\gamma}} + (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\gamma}} - 1 \right] \tag{11}
\end{aligned}$$

Now

$$\begin{aligned}
E(\xi \cup \chi) &= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} \right\}} \sum_{X_{\eta}=X} \left[\mu_{\xi \cup \chi}(x_i) e^{(\mu_{\xi \cup \chi}(x_i))^{\beta}} + (1 - \mu_{\xi \cup \chi}(x_i)) e^{[1 - \mu_{\xi \cup \chi}(x_i)]^{\beta}} - 1 \right] - \\
& \sum_{X_{\theta}=X} \left[\mu_{\xi \cup \chi}(x_i) e^{(\mu_{\xi \cup \chi}(x_i))^{\gamma}} + \left((1 - \mu_{\xi \cup \chi}(x_i)) e^{[1 - \mu_{\xi \cup \chi}(x_i)]^{\gamma}} - 1 \right) \right] \\
&= \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} \right\}} \sum_{X_{\eta}=X} \left\{ \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} + (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\gamma}} - (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\gamma}} \right\} \\
&+ \sum_{X_{\theta}=X} \left\{ \mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\beta}} + (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\beta}} - \mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\gamma}} - (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\gamma}} \right\}
\end{aligned}$$

Therefore

$$E(\zeta \cup (\xi \cup \chi)) = \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} - 1 \right\}} \sum_{X_{\eta}=X} \left\{ \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\alpha}} + (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\alpha}} - \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\beta}} - (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\beta}} - 1 \right\}$$

$$\begin{aligned}
& + \left\{ \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} + (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\gamma}} - (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\gamma}} \right\} + \\
& \sum_{x_{\theta}=x} \left\{ \mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\beta}} + (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\beta}} - \mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\gamma}} - (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\gamma}} \right\} \\
& = \frac{1}{n \left\{ \left[e^{\frac{1}{2}} \right]^{\alpha} - \left[e^{\frac{1}{2}} \right]^{\beta} - \left[e^{\frac{1}{2}} \right]^{\gamma} - 1 \right\}} \sum_{x_{\eta}=x} \left\{ \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\alpha}} + (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\alpha}} - \mu_{\zeta}(x_i) e^{(\mu_{\zeta}(x_i))^{\beta}} - (1 - \mu_{\zeta}(x_i)) e^{(1 - \mu_{\zeta}(x_i))^{\beta}} \right\} + \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\alpha}} + (1 - \\
& \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\alpha}} - \mu_{\xi}(x_i) e^{(\mu_{\xi}(x_i))^{\beta}} - (1 - \mu_{\xi}(x_i)) e^{(1 - \mu_{\xi}(x_i))^{\beta}} \Big\} + \sum_{x_{\theta}=x} \left[\mu_{\chi}(x_i) e^{(\mu_{\chi}(x_i))^{\gamma}} + (1 - \mu_{\chi}(x_i)) e^{(1 - \mu_{\chi}(x_i))^{\gamma}} - 1 \right] \quad (12)
\end{aligned}$$

From equation (11) and (12), we get $E((\zeta \cup \xi) \cup \chi) = E(\zeta \cup (\xi \cup \chi))$

(II) From the above result and from equation (10), we easily see that $E((\zeta \cap \xi) \cap \chi) = E(\zeta \cap (\xi \cap \chi))$

Hence the result

Conclusion

An exponential parametric entropy measure on fuzzy sets is presented in this paper. When the appropriate value of parameters is substituted, the proposed entropy measure, which reduces to the current entropy measure, is significant and valid. These entropy metrics' significant and intriguing characteristics have also been examined. So, In terms of applicability, we may say that the defined entropy measure is more adaptable.

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