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Transformation of series ${}_2\Psi_2$, ${}_3\Psi_3$ and ${}_4\Psi_4$

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Abstract

In this paper we establish some identities involving bilateral hypergeometric series by the use of Gauss Sum and Bailey's transformation formula. We obtain some special cases of these identities.

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Keywords: Transformation, mathematics subject, hypergeometric series, bilateral hypergeometric series

Introduction

The term "hypergeometric series" was first used by J. Wallis in 1656 to refer to a generalization of geometric series. Many mathematicians of the 18th and 19th centuries such as Euler, Gauss [8], Heine [9], Jacobi [13] and Klein [14] contributed to study of hypergeometric series. Hypergeometric series are very important in Mathematics. Many of the familiar functions of analysis are hypergeometric series, these includes the exponential, logarithm, trigonometric, binomial and Bessel functions, along with the classical polynomial sequences of Legendre, Chebychev, Laguerre, Hermite etc. Bilateral hypergeometric series have connection to various combinatorial identities including those related to partition and q -series. Also used to derive and prove identities involving eta function, gamma function, beta function etc. Bilateral hypergeometric series are powerful tools with a wide range of applications from solving differential equations to exploring combinatorial identities and developing special functions. For more details about Hypergeometric series one can see [2, 1, 3, 77, 11, 12, 20, 16, [19, 18, 17].

Throughout the paper, we adopt the following definitions and notations

For a and q are complex parameters with $|q| < 1$, the q -shifted factorial is defined as:

$$\begin{aligned}(a; q)_0 &= 1 \\ (a; q)_n &= \prod_{k=1}^n (1 - aq^{k-1}), n \in \mathbb{N} \\ (a; q)_\infty &= \prod_{k=0}^{\infty} (1 - aq^k)\end{aligned}$$

The following notations are also used in our investigation

$$\begin{aligned}(a_1; q)_n (a_2; q)_n (a_3; q)_n \dots (a_r; q)_n &= (a_1, a_2, \dots, a_r; q)_n \\ (a_1; q)_\infty (a_2; q)_\infty (a_3; q)_\infty \dots (a_r; q)_\infty &= (a_1, a_2, \dots, a_r; q)_\infty.\end{aligned}$$

Gaspar and Rahman [7] defined the basic hypergeometric series as;

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} z^n \{(-1)^n q^{n(n-1)/2}\}^{1+s-r} \quad (1.1)$$

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Which converges for $|z| < \infty$ if $r \leq s$ and for $|z| < 1$ if $r = s + 1$.

The basic bilateral hypergeometric series is defined as

$${}_r\psi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} z^n \{(-1)^n q^{n(n-1)/2}\}^{s-r} \quad (1.2)$$

Which converges for $\left| \frac{b_1 b_2 b_3 \dots b_s}{a_1 a_2 a_3 \dots a_r} \right| < |z| < 1$ if $r = s$ and for $s > r$ it converges in the whole complex plane i.e for $|z| < \infty$.

From [7], the Bailey's transform is given by

$${}_2\Psi_2 \left[\begin{matrix} a, b \\ c, d \end{matrix}; q, z \right] = \frac{(az, bz, cq/abz, dq/abz; q)_{\infty}}{(q/a, q/b, c, d; q)_{\infty}} \times {}_2\Psi_2 [abz/c, abz/d, az, bz; q; cd/abz] \quad (1.3)$$

The q-Gauss sum [7] for ${}_2\phi_1$ series is given by

$${}_2\phi_1(a, b; c; q, z) = \frac{(c/a, c/b; q)_{\infty}}{(c, c/ab; q)_{\infty}} \quad (1.4)$$

Where $|z| < 1$ and $|b| < 1$.

2. New q-series identities

In this Section, we derive new q-series identities employing Bailey's transformation formula [7, (5.20)(i), p.137].

Theorem 2.1. We have

$${}_2\Psi_2[a, a, aq, d; q; q/a] = \frac{(q; q)_{\infty}^3 (d/a; q)_{\infty}}{(q/a; q)_{\infty}^2 (d; q)_{\infty} (aq; q)_{\infty}} \quad (2.1)$$

And

$${}_2\Psi_2[a, a, aq, d; q; 1/a] = a \frac{(q; q)_{\infty}^3 (d/a; q)_{\infty}}{(q/a; q)_{\infty}^2 (d; q)_{\infty} (aq; q)_{\infty}} \quad (2.2)$$

Where a, d, q are complex parameters with $|q| < 1$.

Proof. Setting $b = a, c = aq$ and $z = q/a$ in (1.3), we get

$${}_2\Psi_2[a, a, aq, d; q; q/a] = \frac{(q, d/a, q, d/a; q)_{\infty}}{(q/a, d, q/a, d; q)_{\infty}} \times {}_2\Psi_2[a, aq/d, q, aq; q; d/a]$$

Which implies

$${}_2\Psi_2[a, a, aq, d; q; q/a] = \frac{(q, d/a, q, d/a; q)_{\infty}}{(q/a, d, q/a, d; q)_{\infty}} \times {}_2\Phi_1[a, aq/d, aq; q; d/a] \quad (2.3)$$

Using q-Gauss sum (1.4) in (2.3), we have

$${}_2\Phi_1[a, aq/d, aq; q; d/a] = \frac{(q, d; q)_{\infty}}{(aq, d/a; q)_{\infty}} \quad (2.4)$$

Employing (2.4) in (2.3), we have

$${}_2\Psi_2[a, a, aq, d; q; q/a] = \frac{(q; q)_{\infty}^3 (d/a; q)_{\infty}}{(q/a; q)_{\infty}^2 (d; q)_{\infty} (aq; q)_{\infty}} \quad (2.5)$$

Which is the required result (2.1).

Taking

$${}_2\Psi_2[a, a, aq, d; q; 1/a] - a {}_2\Psi_2[a, a, aq, d; q; q/a] = (1 - a) {}_1\Psi_1[a, d; q; 1/a]$$

Which can be written as

$${}_2\Psi_2[a, a, aq, d; q; 1/a] = (1 - a) {}_1\Psi_1[a, d; q; 1/a] + a {}_2\Psi_2[a, a, aq, d; q; q/a] \quad (2.6)$$

Using the Ramanujan's sum [?] i.e ${}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_{\infty}}{(b, q/a, z, b/az; q)_{\infty}}$, equation 2.6 becomes

$${}_2\Psi_2[a, a, aq, d; q; 1/a] = a {}_2\Psi_2[a, a, aq, d; q; q/a] \quad (2.7)$$

From (2.5) and (2.7), we get

$${}_2\Psi_2[a, a, aq, d; q; 1/a] = a \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty}$$

Which is the result (2.2)

Theorem 2.2. We have

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] = \frac{(a-c)}{(1-c)} {}_2\Psi_2[a, a, aq, d; q; q/a] \quad (2.8)$$

Or

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] = \frac{(a-c)}{(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.9)$$

And

$${}_3\Psi_3[a, b, cq, bq, d, c; q; q/a] = \frac{(a-c)}{a(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.10)$$

Where a, d, c, q are complex parameters with $|q| < 1$.

Proof. Now, considering

$${}_3\Psi_3 \left[\begin{matrix} a, a, c \\ d, aq, cq \end{matrix}; q, 1/a \right] - c {}_3\Psi_3 \left[\begin{matrix} a, a, c \\ d, aq, cq \end{matrix}; q, q/a \right] = (1-c) {}_2\Psi_2[a, a, aq, d; q; 1/a] \quad (2.11)$$

From equations (2.11) and (2.8), we have

$${}_3\Psi_3 \left[\begin{matrix} a, a, c \\ d, aq, cq \end{matrix}; q, 1/a \right] - c {}_3\Psi_3 \left[\begin{matrix} a, a, c \\ d, aq, cq \end{matrix}; q, q/a \right] = (1-c) a \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.12)$$

Again, considering

$$= \sum_{n=-\infty}^{\infty} \frac{{}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a]}{(d, aq; q)_n} \frac{1}{a^n} \frac{(1-cq^n)}{(1-c)} = \frac{1}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} a, a \\ aq, d \end{matrix}; q, 1/a \right] \quad (2.13)$$

and involving equation (2.7) in (2.13), we get

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] = \frac{(a-c)}{(1-c)} \times {}_2\Psi_2[a, a, aq, d; q; q/a] \quad (2.14)$$

Using equation (2.5) in (2.14), we have

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] = \frac{(a-c)}{(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.15)$$

Which is the result (2.9).

Now, let us consider

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] - a {}_3\Psi_3[a, a, cq, aq, d, c; q; q/a] \\ = (1-a) {}_2\Psi_2[a, cq, d, c; q; 1/a]$$

Summing ${}_1\Psi_1$ series in equation (2.16), we obtain

$${}_3\Psi_3[a, a, cq, aq, d, c; q; 1/a] = a {}_3\Psi_3[a, a, cq, aq, d, c; q; q/a] \quad (2.17)$$

From (2.15) and (2.17), we have

$${}_3\Psi_3[a, b, cq, bq, d, c; q; q/a] = \frac{(a-c)}{a(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.18)$$

Theorem 2.3. We have

$${}_4\Psi_4[a, a, cq, \lambda q, aq, d, c, \lambda; q; 1/a] = \frac{(a-c)(a-\lambda)}{a(1-\lambda)(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.19)$$

Where a, d, c, q are complex parameters with $|q| < 1$

Proof. Now, taking

$${}_4\Psi_4[a, a, cq, \lambda q, aq, d, c, \lambda; q; 1/a] = \sum_{n=-\infty}^{\infty} \frac{(a, a, cq; q)_\infty (1-\lambda q^n)}{(d, aq, c; q)_\infty (1-\lambda)} \frac{1}{a^n} \quad (2.20)$$

Using (2.9) and (2.18) in (2.20), we have

$${}_4\Psi_4[a, a, cq, \lambda q, aq, d, c, \lambda; q; 1/a] = \frac{(a-c)(a-\lambda)}{a(1-\lambda)(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (2.21)$$

This is the proof of (2.19).

3. Special Cases

In this Section, we derive some special cases of our Theorem 2.3.

1. Taking $c = 0$ in 2.19, we have

$${}_3\Psi_3\left[\begin{matrix} a, a, \lambda q \\ aq, d, \lambda \end{matrix}; q, 1/a\right] = \frac{(a-\lambda)}{(1-\lambda)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (3.1)$$

2. Taking $\lambda = 0$ in (2.19), we have

$${}_3\Psi_3\left[\begin{matrix} a, b, cq \\ aq, d, c \end{matrix}; q, 1/a\right] = \frac{(a-c)}{(1-c)} \frac{(q; q)_\infty^3 (d/a; q)_\infty}{(q/a; q)_\infty^2 (d; q)_\infty (aq; q)_\infty} \quad (3.2)$$

3. Taking $\lambda = 0$ and $d = 0$ in (2.19), we have

$${}_3\Psi_2\left[\begin{matrix} a, a, cq \\ aq, c \end{matrix}; q, 1/a\right] = \frac{(a-c)}{(1-c)} \frac{(q; q)_\infty^3}{(q/a; q)_\infty^2 (aq; q)_\infty} \quad (3.3)$$

4. Taking $d = q$ and $c = 0$ in 2.19, we have

$${}_3\Psi_3\left[\begin{matrix} a, a, \lambda q; q, 1/a \\ aq, q, \lambda \end{matrix}\right] = \frac{(a-\lambda)}{(1-\lambda)} \frac{(q; q)_\infty^2}{(q/a; q)_\infty (aq; q)_\infty} \quad (3.4)$$

5. Taking $d = c = \lambda = 0$ in (2.19), we have

$${}_2\Phi_1\left[\begin{matrix} a, a \\ aq \end{matrix}; q, 1/a\right] = a \frac{(q; q)_\infty^3}{(q/a; q)_\infty^2 (aq; q)_\infty} \quad (3.5)$$

Conclusion

Hypergeometric series, pioneered by Wallis and extensively studied by mathematicians like Euler, Gauss, and Klein, are fundamental in analysis and numerous mathematical applications. Their significance spans from representing classical functions to deriving identities in combinatorics, special functions, and differential equations. Bilateral hypergeometric series, in particular, provide powerful tools for understanding complex relationships among functions such as the eta, gamma, and beta functions. This paper introduced new q-series identities through Bailey's transformation, enriching the theory and offering potential for further exploration in mathematical analysis and related fields.

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