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Love waves in heterogeneous transversely isotropic media

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Abstract

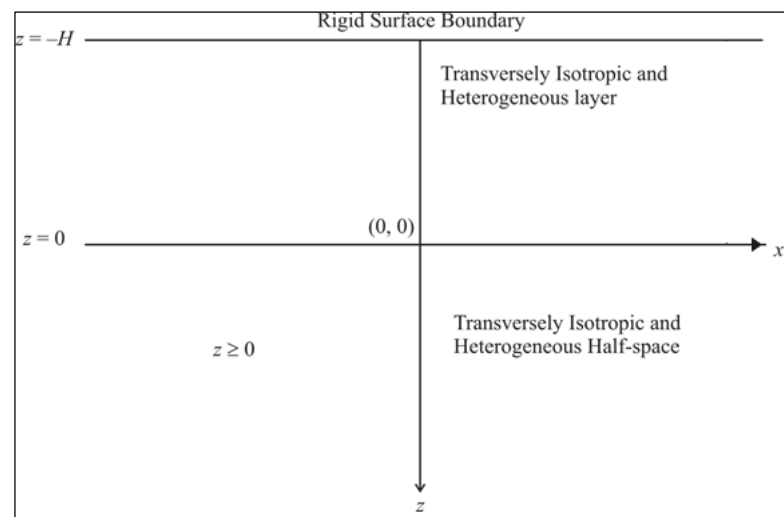
A problem of Love type of surface wave propagation is considered in a laterally and vertically inhomogeneous transversely isotropic elastic media consisting of a layer lying over a semi-infinite half-space. The frequency equation is derived when the boundary of the upper surface of the layer is loaded by some rigid structure such that here the displacement is zero and the stresses do not vanish. The restriction imposed for the existence of the real roots of the frequency equation on phase velocity is obtained.

Keywords: Laterally and vertically inhomogeneous rigid structure, Phase velocity, Frequency equation, Isotropic

Introduction

Love wave propagation plays a vital role to know the internal structure of the earth and somehow its possible application in the prediction of earthquakes. Anderson (1962)^[1], Haskell (1962)^[3] studied the wave propagation in multilayered elastic media. Gogna (1979)^[2] has studied the propagation of a surface wave in a media having a homogeneous anisotropic layer lying over a homogeneous isotropic half-space and under a uniform liquid layer. Various authors S. Manna *et al.* (2015)^[4], S. Kundu *et al.* (2016)^[5] and S. Bhat and S. Manna (2024)^[6] have studied on wave propagation choosing the models of different media.

Idea here, is to examine the propagation of Love wave in a medium occupying a half-space under a layer of laterally and vertically inhomogeneous transversely isotropic elastic media. The upper surface of the layer is taken rigid boundary. The plane $z = 0$ separates the medium layer from the half-space ($z \geq 0$). The geometry of the problem is shown in Fig. 1.



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Fig 1

Basic equations and solutions

The strain energy volume density function W for transversely isotropic elastic medium is given by Love (1944)

$$2W = A(e_{xx}^2 + e_{yy}^2) + Ce_{zz}^2 + 2F(e_{yy} + e_{xx})e_{zz} + 2(A - 2N)e_{xx}e_{yy} + L(e_{yz}^2 + e_{zx}^2) + Ne_{xy}^2, \quad (2.1)$$

where A, C, F, N and L are elastic parameters, e_{xx}, e_{yz} etc. are the components of strain at a given point. The components of stress τ_{ij} are given by

$$\tau_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad (i, j = x, y, z) \quad (2.2)$$

We consider a two-dimensional motion in xz -plane and assume that all quantities are independent of the y -coordinate, i.e., $\frac{\partial}{\partial y} = 0$. In this case the equation of motion

$$\tau_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (2.3)$$

describing SH-type of motion for which $u_1 = u_3 = 0, u_2 = v$ reduces to

$$\frac{\partial}{\partial x} \left(N \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial z} \left(L \frac{\partial v}{\partial z} \right) = \rho \frac{\partial^2 v}{\partial t^2} \quad (2.4)$$

where N and L are directional rigidities and ρ is the density of the medium.

To get a time harmonic solution of (2.4), we assume that

$$v(x, z, t) = V(x, z)e^{-i\omega t} \quad (2.5)$$

where ω is the angular frequency.

Further, if we put

$$V = L^{-\frac{1}{2}}(x, z)D(x, z) \quad (2.6)$$

The equation (2.4) then takes the form

$$\begin{aligned} \frac{\partial^2 D}{\partial x^2} + \frac{L}{N} \frac{\partial^2 D}{\partial z^2} + \left[\frac{1}{N} \frac{\partial N}{\partial x} - \frac{1}{L} \frac{\partial L}{\partial x} \right] \frac{\partial D}{\partial x} + \left[\frac{3}{4} \frac{1}{L^2} \left(\frac{\partial L}{\partial x} \right)^2 - \frac{1}{2L} \frac{\partial^2 L}{\partial x^2} - \frac{1}{2LN} \frac{\partial L}{\partial x} \frac{\partial N}{\partial x} \right] D \\ + \frac{1}{N} \left[\frac{1}{4L^2} \left(\frac{\partial L}{\partial x} \right)^2 - \frac{1}{2L} \frac{\partial^2 L}{\partial z^2} \right] D + \frac{\rho}{N} \omega^2 D = 0 \end{aligned} \quad (2.7)$$

We denote $\beta_{11} = \left(\frac{N}{\rho} \right)^{\frac{1}{2}}$ and $\beta_1 = \left(\frac{1}{\rho} \right)^{\frac{1}{2}}$ as SH-wave velocities in horizontal and in vertical directions respectively. Further L ,

N and ρ are assumed such functions of x and z that β_{11} and β_1 are constants. Thus, we assume that

$$N(x, z) = N_0 p(x) r(z),$$

$$L(x, z) = L_0 p(x) r(z),$$

$$\rho(x, z) = \rho_0 p(x) r(z) \quad (2.8)$$

where N_0 , L_0 and ρ_0 are constants, equation (2.7) becomes

$$\frac{\partial^2 D}{\partial x^2} + \frac{L_0}{N_0} \frac{\partial^2 D}{\partial z^2} + \left[-\varepsilon^2 - b^2 \frac{L_0}{N_0} + \frac{\rho_0}{N_0} \omega^2 \right] D = 0 \quad (2.9)$$

where, r and p satisfy the following differential equations

$$\frac{1}{4} \left(\frac{r'}{r} \right)^2 - \frac{1}{2} \frac{r''}{r} = -b^2 \quad (2.10)$$

$$\frac{1}{4} \left(\frac{p'}{p} \right)^2 - \frac{1}{2} \frac{p''}{p} = -\varepsilon^2 \quad (2.11)$$

in which b and ε are real constants.

Further for waves propagating in positive x -direction, we take the solution of equation (2.9) in the form

$$D(x, z) = Z(z) e^{ikx} \quad (2.12)$$

where $Z(z)$ must satisfy

$$\frac{d^2 Z}{dz^2} - \frac{N_0}{L_0} \left(\varepsilon^2 + k^2 + \frac{L_0}{N_0} b^2 - \frac{\omega^2}{\beta_{110}^2} \right) Z = 0 \quad (2.13)$$

in which

$$\beta_{110} = \left(\frac{N_0}{\rho_0} \right)^{\frac{1}{2}}, \text{ and } k \text{ is the wave number.}$$

The solution of equation (2.12) can be written as

$$Z = A e^{S_1 z} + B e^{-S_1 z} \quad (2.14)$$

where A and B are constants and S_1 is given by

$$S_1 = \left(\frac{N_0}{L_0} \right)^{\frac{1}{2}} \left(\varepsilon^2 + k^2 + \frac{L_0}{N_0} b^2 - \frac{\omega^2}{\beta_{110}^2} \right)^{\frac{1}{2}}, \text{ if } \frac{\omega}{\beta_{110}} < \left(\varepsilon^2 + k^2 + \frac{L_0}{N_0} b^2 \right)^{\frac{1}{2}},$$

$$-i \left(\frac{N_0}{L_0} \right)^{\frac{1}{2}} \left(\frac{\omega^2}{\beta_{110}^2} - \varepsilon^2 - k^2 - \frac{L_0}{N_0} b^2 \right)^{\frac{1}{2}}, \text{ if } \frac{\omega}{\beta_{110}} > \left(\varepsilon^2 + k^2 + \frac{L_0}{N_0} b^2 \right)^{\frac{1}{2}} \quad (2.15)$$

Thus, the displacement component $v(x, z, t)$ is given by

$$v(x, z, t) = \frac{1}{[L_0 p(x) r(z)]^{\frac{1}{2}}} [A e^{S_1 z} + B e^{-S_1 z}] e^{ik(x-ct)} \quad (2.16)$$

where, $\omega = ck$.

And the only non-zero stress component τ_{yz} , is

$$\tau_{yz} = L e_{yz},$$

$$= L \frac{\partial v}{\partial z},$$

$$= \left[\left\{ L_0 p(x) r(z) \right\}^{\frac{1}{2}} S_1 \left(A e^{S_1 z} - B e^{-S_1 z} \right) - \frac{1}{2} \frac{L_0 p(x) r'(z)}{\left[L_0 p(x) r(z) \right]^{\frac{1}{2}}} \left(A e^{S_1 z} + B e^{-S_1 z} \right) \right] e^{ik(x-ct)} \quad (2.17)$$

So, the displacement and stresses in the layer are given by (2.16) and (2.17); and in the half-space the displacement defined by v_1 and stress components defined by $(\tau_{yz})_1$ may be written in the form

$$V_1(x, z, t) = \frac{1}{\left[L_{01} p(x) \ell(z) \right]^{\frac{1}{2}}} A_1 e^{-S_2 z} e^{ik(x-ct)} \quad (2.18)$$

where for surface waves, S_2 is given by

$$S_2 = \left(\frac{N_{01}}{L_{01}} \right)^{\frac{1}{2}} \left[\varepsilon^2 + k^2 + \frac{L_{01}}{N_{01}} a^2 - \frac{\omega^2}{\beta_2^2} \right] \quad (2.19)$$

which is positive and real quantity,
and

$$\beta_2 = \left(\frac{N_{01}}{\rho_{01}} \right)^{\frac{1}{2}}.$$

$$(\tau_{yz})_1 = L_1 \left(\frac{\partial v_1}{\partial z} \right) = - \left[\left\{ L_{01} p(x) \ell(z) \right\}^{\frac{1}{2}} S^2 + \frac{1}{2} \frac{L_{01} p(x) \ell'(z)}{\left[L_{01} p(x) \ell(z) \right]^{\frac{1}{2}}} \right] A_1 e^{-S_2 z} e^{ik(x-ct)} \quad (2.20)$$

where in the half-space the directional rigidities and density are assumed to vary as

$$\begin{aligned} N &= N_{01} p(x) \ell(z), \\ L &= L_{01} p(x) \ell(z), \\ \rho &= \rho_{01} p(x) \ell(z) \end{aligned} \quad (2.21)$$

where $\ell(z)$ satisfy the differential equation similar to (2.10) given by

$$\frac{1}{4} \left(\frac{\ell'}{\ell} \right)^2 - \frac{1}{2} \frac{\ell''}{\ell} = -a^2 \quad (2.22)$$

3. Boundary conditions and the frequency equations

Rigid boundary on the plane $Z = -H$

Since there is no slip at interface $Z = 0$ between elastic layer and the half-space and no displacement at the rigid boundary $Z = -H$. So the boundary conditions may be written as

$$\begin{aligned} \text{(i)} \quad & v = 0 \quad \text{at } Z = -H, \\ \text{(ii)} \quad & v = v_1 \quad \text{at } Z = 0, \\ \text{(iii)} \quad & \tau_{yz} = (\tau_{yz})_1 \quad \text{at } Z = 0 \end{aligned} \quad (3.1)$$

Using the relations (2.16) to (2.20), the above boundary conditions lead to the following homogeneous system of linear equations in A, B and A_1

$$Ae^{-S_1 H} + Be^{S_1 H} = 0,$$

$$\begin{aligned} & \sqrt{\frac{L_{01}}{L_0} \left(\frac{\ell(0)}{r(0)} \right)} (A + B) - A_1 = 0 \\ & \left[S_1 - \frac{1}{2} \frac{r'(0)}{r(0)} \right] A - \left[S_1 + \frac{1}{2} \frac{r'(0)}{r(0)} \right] B \\ & + \sqrt{\frac{L_{01}}{L_0} \frac{\ell(0)}{r(0)}} \left(S_2 + \frac{1}{2} \frac{\ell'(0)}{\ell(0)} \right) A_1 = 0 \end{aligned} \quad (3.2)$$

For non-zero solutions of (3.2), we have

$$\det(a_{ij}) = 0 \quad (3.3)$$

where a_{ij} is an 3×3 matrix whose elements are

$$a_{11} = e^{-S_1 H}, \quad a_{12} = e^{S_1 H}, \quad a_{13} = 0,$$

$$a_{21} = a_{22} = \left[\frac{L_{01}}{L_0} \frac{\ell(0)}{r(0)} \right]^{\frac{1}{2}}, \quad a_{23} = -1,$$

$$a_{31} = S_1 - \frac{1}{2} \frac{r'(0)}{r(0)}, \quad a_{32} = - \left[S_1 + \frac{1}{2} \frac{r'(0)}{r(0)} \right],$$

$$a_{33} = \sqrt{\frac{L_{01}}{L_0} \frac{\ell(0)}{r(0)}} \left(S_2 + \frac{1}{2} \frac{\ell'(0)}{\ell(0)} \right).$$

On expansion equation (3.3) becomes

$$\tanh(S_1 H) = \frac{S_1}{\frac{1}{2} \frac{r'(0)}{r(0)} - \frac{L_{01}}{L_0} \frac{\ell(0)}{r(0)} \left(S_2 + \frac{1}{2} \frac{\ell'(0)}{\ell(0)} \right)} \quad (3.4)$$

For the real roots of this equation S_1 should be purely imaginary, so taking $S_1 = -iS$ and S_2 should be real and positive (3.4) become

$$\tan(SH) = \frac{S}{\frac{1}{2} \frac{r'(0)}{r(0)} - \frac{L_{01}}{L_0} \frac{\ell(0)}{r(0)} \left[S_2 + \frac{1}{2} \frac{\ell'(0)}{\ell(0)} \right]} \quad (3.5)$$

where

$$S = \left(\frac{N_0}{L_0} \right)^{\frac{1}{2}} \left[\frac{\omega^2}{\beta_{110}^2} - \varepsilon^2 - k^2 - \frac{L_0}{N_0} b^2 \right]^{\frac{1}{2}},$$

$$S_2 = \left(\frac{N_{01}}{L_{01}} \right)^{\frac{1}{2}} \left[\varepsilon^2 + k^2 + \frac{L_{01}}{N_{01}} a^2 - \frac{\omega^2}{\beta_2^2} \right]^{\frac{1}{2}},$$

$$\beta_{110} = \left(\frac{N_0}{\rho_0} \right)^{\frac{1}{2}},$$

$$\beta_2 = \left(\frac{N_{01}}{\rho_{01}} \right)^{\frac{1}{2}} \quad (3.6)$$

Equation (3.5) is the frequency equation.

The conditions that S_2 be positive real, and S_1 purely negative and imaginary impose a restriction given by

$$\beta_{110} \left[\frac{\varepsilon^2}{k^2} + 1 + \frac{L_0}{N_0} \frac{b^2}{k^2} \right]^{\frac{1}{2}} < c < \beta_2 \left[\frac{\varepsilon^2}{k^2} + 1 + \frac{L_{01}}{N_{01}} \frac{a^2}{k^2} \right]^{\frac{1}{2}} \quad (3.7)$$

on the phase velocity.

4. Conclusion

Discussed the condition for the existence of real roots of the frequency equation.

The condition imposes a restriction on phase velocity as

$$\beta_{110} \left[\frac{\varepsilon^2}{k^2} + 1 + \frac{L_0}{N_0} \frac{b^2}{k^2} \right]^{\frac{1}{2}} < c < \beta_2 \left[\frac{\varepsilon^2}{k^2} + 1 + \frac{L_{01}}{N_{01}} \frac{a^2}{k^2} \right]^{\frac{1}{2}}$$

5. References

1. Anderson DL. Love wave dispersion in heterogeneous anisotropic media. *Geophysics*. 1962;27(4):445-454.
2. Gogna ML. Surface wave propagation in a homogeneous anisotropic layer lying over a homogeneous semi-infinite isotropic half-space under a uniform layer of liquid. *Bull Indian Soc Earthq Technol*. 1979;16:1-12.
3. Haskell NA. Crystal reflection of plane P and SV waves. *J Geophys Res*. 1962;67:4751-4769.
4. Manna S, Kundu S, Gupta S. Love wave propagation in a piezoelectric layer overlying an inhomogeneous elastic half-space. *J Vib Control*. 2015;21(13):2553-2568.
5. Kundu S, Gupta S, Vaishnav PK, Manna S. Propagation of Love wave in a heterogeneous medium over an inhomogeneous half-space under the effect of point source. *J Vib Control*. 2016;22(5):1325-1337.
6. Bhat S, Manna S. Rayleigh wave fields in a multilayered micro media. *Int J Geomech*. 2024;24(4):04024018.