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Jump diffusion geometric Brownian motion with the re-invested dividend yielding asset and transaction cost

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Abstract

We consider an investment model where surplus evolves under a jump diffusion risk process. The financial institution reinvests proportional dividends and allocates surplus between a risk-free asset and a risky asset whose return follows a jump diffusion process. While jump diffusion models capture discontinuities in asset pricing, they often have limitations. To enhance realism, we model the risky asset's price using a Brownian motion risk model incorporating dividends and transaction costs, where the instantaneous investment return follows a Geometric Brownian motion with jump diffusion. Applying Itô's lemma, we derive the asset price for this framework, accounting for reinvested dividends and transaction costs.

Keywords: Geometric brownian motion, jump diffusion, dividend yielding asset, stochastic process, wiener process, Itô's process

1. Introduction

The Black-Scholes model, developed by Fischer Black and Myron Scholes in 1973 ^[1], is a fundamental tool in financial mathematics for pricing options. This model revolutionized the options market by providing a systematic method for valuing European-style options. The Black-Scholes framework consists of two primary assets: a riskless bond with a constant interest rate and a stock whose price follows a geometric Brownian motion (GBM). The stock price dynamics are given by:

$$S(t) = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right), \quad S_0 > 0 \quad (1)$$

Where $S(t)$ represents the stock price at time t , S_0 is the initial stock price, μ is the drift term (constant rate of return), σ is the volatility, and B_t is the standard Brownian motion ^[2]. The standard Black-Scholes model relies on several key assumptions:

- Constant Volatility-The model assumes the asset's price volatility remains unchanged over time.
- Efficient Markets-No arbitrage opportunities exist.
- No Dividends-The model does not account for dividend payments.
- Constant Risk-Free Interest Rate-The risk-free rate is assumed to be fixed.
- Log-Normal Returns-The underlying asset's returns follow a log-normal distribution.
- European-Style Options-Options can only be exercised at expiration.
- No Transaction Costs-Trading occurs without commissions or costs.
- Continuous Trading-The model assumes uninterrupted market activity.

While these assumptions simplify the mathematical formulation, they do not fully capture the complexities of real-world markets. Various modifications, such as stochastic volatility models and jump diffusion models, have been developed to address these limitations.

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A crucial component of the Black-Scholes model is the geometric Brownian motion (GBM), which governs stock price evolution. The differential equation for GBM is given by:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (2)$$

where $W(t)$ is the Wiener process representing the random fluctuations in stock price dynamics. The equation consists of two components:

1. Deterministic Growth ($\mu S(t)dt$)-Models the expected return of the stock.
2. Random Fluctuations ($\sigma S(t) dW(t)$)-Captures stochastic variations due to market randomness.

Despite its utility, empirical studies suggest that stock prices may exhibit sudden jumps due to unexpected economic events, financial announcements, or external shocks. The geometric Brownian motion fails to account for these discontinuous movements, leading to the development of more advanced models, such as the jump diffusion model introduced by Merton. This model extends the standard GBM by incorporating discrete jumps in asset prices, allowing for a more accurate representation of real-world financial markets^[3].

Our current model, A Jump Diffusion Geometric Brownian Motion with the Re-Invested Dividend Yielding Asset and Transaction Cost, expands upon these advancements by introducing an additional layer of realism to asset price modeling. Unlike traditional jump diffusion models, our approach integrates:

1. Re-Invested Dividend Yields-Captures the compounding effect of reinvesting dividends, influencing asset growth dynamics.
2. Transaction Costs-Incorporates the friction of trading costs, making the model more applicable to real-world financial markets.
3. Jump Diffusion Process-Enhances price modeling by allowing for sudden, stochastic price jumps in asset valuation.

This extended framework provides a more comprehensive tool for pricing and risk assessment by accommodating factors that traditional models overlook. It not only refines derivative pricing but also improves investment decision-making in markets where dividends and trading costs play a significant role^[4].

In conclusion, while the Black-Scholes model remains a cornerstone of financial mathematics, its assumptions limit its applicability in real-world markets. The incorporation of stochastic volatility, jump processes, dividend reinvestments, and transaction costs offers a more robust framework for modeling asset price dynamics and improving derivative pricing strategies.

2. Methodology

In this section, we recall some key basic concepts, methodologies and the result that are fundamental in the study:

2.1. Stochastic Process

A stochastic process can be denoted as $X = [X(t); t \in (0; \alpha)]$ which is a collection of random variables such that for each t in the index set $(0; \alpha)$ which specifies the time interval over which the process is defined, $X(t)$ is a random variable that represents the state of the process at time t . A discrete time stochastic is the one where the value of the variable can only change at a certain fixed points in time. On the other hand, in a continuous time stochastic process, change can take any value within a certain range^[3].

2.2. Markov stochastic process

A Markov stochastic process, or simply a Markov process, is a specific type of stochastic process where the future movements of the process depend solely on its current state.

2.3. Wiener process (Brownian Motion)

The Wiener process $W(t)$, also known as Brownian motion, is a continuous-time stochastic process that has wide applications in various fields, including finance, physics, and statistics. The Wiener process, denoted as $W(t)$ for $t \geq 0$, is characterized by the following properties:

1. Normal increments: The increments of the Wiener process, $W(t) - W(s)$, where $0 \leq s < t$ are normally distributed. Specifically, $W(t) - W(s) \sim N(0, t - s)$ meaning that the increment has a mean of 0 and a variance equal to the time difference $t - s$.
2. Independent increments: The increments of the Wiener process are independent of each other. This property states that for disjoint time intervals, the increments are statistically independent. For example, $W(t_1) - W(s_1)$ and $W(t_2) - W(s_2)$ are independent when $t_1 < s_1 < t_2 < s_2$. This independence property allows for the modeling and analysis of complex systems involving random processes.
3. Continuity of paths: The Wiener process has almost surely continuous paths. This means that the sample paths of the process are continuous functions of time with probability 1. The continuous paths property distinguishes the Wiener process from other stochastic processes that exhibit jumps or discontinuities. i.e $W(t)$ is a continuous function of t .

where s and t are functions of time^[3].

2.4. Generalised Wiener Process

The Generalized Wiener process has continuous sample paths, like the standard Wiener process, making it suitable for modeling continuous phenomena. However, its long-range dependence property allows it to capture complex behaviors and trends that

standard Brownian motion cannot represent. A generalised Wiener process defines the dynamics of X in terms of a deterministic component adt and a stochastic component bdW as;

$$dX = adt + bdW, \quad (3)$$

where a represents the mean rate or drift of the variable X and b represents the variance rate or diffusivity of X . It quantifies the magnitude of the random fluctuations or noise added to the process, dt is the infinitesimal time interval (often referred to as an "infinitesimal change in time") and dW represents the Wiener process or Brownian motion, which introduces randomness or variability to the path followed by X .

By incorporating both the deterministic and stochastic components, the generalized Wiener process allows for modeling variables that exhibit both expected trends and random fluctuations.

In a small interval Δt , the change in the value of X denoted as ΔX is given by;

$$\Delta X = a\Delta t + b\varepsilon\sqrt{\Delta t} \quad (4)$$

where ε is a random variable drawn from a standardized normal distribution.

Δt represents the sum of the deterministic change $a\Delta t$ and the random noise $b\varepsilon\sqrt{\Delta t}$. Therefore, ΔX follows a normal distribution with mean of

$$a\Delta t \quad (5)$$

and a Standard deviation of

$$\Delta X = b\sqrt{\Delta t} \quad (6)$$

Hence

$$\Delta X \sim N(a\Delta t; b\sqrt{\Delta t}) \quad (7)$$

Similar argument in Wiener process shows that the change in the value of X in any time interval T is normally distributed with mean of change in $X = aT$, Variance of change in $X = b^2T$ and Standard deviation of change in $X = b\sqrt{T}$ Hence

$$\Delta X \sim N(aT; b\sqrt{T}) \quad (8)$$

2.5. Ito's Lemma

Ito's Lemma is a powerful tool in stochastic calculus and is widely used in the derivation of various financial models, such as the Black-Scholes option pricing model. It allows for the analysis and manipulation of functions of stochastic processes, enabling the calculation of derivatives and the development of stochastic differential equations that describe the dynamics of complex systems involving randomness.

By applying Ito's Lemma, one can analyze and model the behavior of financial derivatives, asset prices, and other stochastic processes, taking into account the impact of random fluctuations and incorporating uncertainty into mathematical models. This is the formula used for solving stochastic differential equations. Suppose that the value X follows an Ito's Process, then

$$dX = a(X; t)dt + b(X; t)dW, \quad (9)$$

where dW is a Wiener process, a and b are functions of X and t . The variable X has a drift rate of a and variance of b^2 . Ito's Lemma shows that the function $G(X; t)$ twice differentiable in X and once in t , is also an Ito's process given by;

$$dG = \left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial X^2 b^2} \right) dt + \frac{\partial G}{\partial X} b dW, \quad (10)$$

where dW is the Wiener process thus G also follows an Ito's process with a drift rate of;

$$\left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial X^2 b^2}\right) \quad (11)$$

and a variance rate of;

$$\left(\frac{\partial G}{\partial X}\right)^2 b^2 \quad (12)$$

2.6. Geometric Brownian motion

A geometric Brownian motion (GBM) is a continuous-time stochastic process that is commonly used to model the price dynamics of financial assets such as stocks, currencies, and commodities. It is an extension of the standard Brownian motion (also known as the Wiener process) that incorporates exponential growth and volatility. A specific type of Ito's process is the geometric brownian motion of the form;

$$dX = aXdt + bXdW, \quad (13)$$

where $a(X, t) = aX$ and $b(X, t) = bX$ ^[3] in the equation above, the geometric Brownian motion has been applied in stock pricing and is given as;

$$dS_t = \mu S_t dt + \sigma S_t dW, \quad (14)$$

where S_t represents the value of the asset at time t , μ is the average rate of return or drift of the asset per unit time and σ is the standard deviation of the asset's returns or volatility per unit time and $dW(t)$ is a Wiener process increment, representing a random shock or noise in the system. The term $\mu S_t dt$ represents the deterministic growth component, which accounts for the expected return or trend in the stock price while the term $\sigma S_t dW(t)$ represents the random or stochastic component which is scaled by the stock price S_t and the volatility σ . The equation is expressed as;

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW, \quad (15)$$

The solution to this stochastic differential equation is given by the following exponential process:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \quad (16)$$

where S_0 is the initial value of the asset at time $t = 0$, and $W(t)$ represents the Wiener process.

3. Results

To study and derive the price of a jump diffusion Geometric Brownian motion with the re-invested dividend and transaction cost, Geometric Brownian motion of the stock price, Geometric Brownian motion with jump process, Geometric Brownian motion with dividend yielding asset with transaction cost was considered.

3.1 Derivation of Stock price following Geometric Brownian motion

Ito's Lemma is a fundamental tool in stochastic calculus used to describe the dynamics of a stochastic process like stock price following Geometric Brownian motion.

Ito's lemma is given by the following formula;

$$dY = \left(\frac{\partial Y}{\partial t} + \mu S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}\right)dt + \sigma S \frac{\partial Y}{\partial S}dW \quad (17)$$

To derive the Geometric Brownian motion using Ito's Lemma;

$$dS = \mu S dt + \sigma S dW \quad (18)$$

From equation 18, we want to find dY where Y is defined as $\ln(S_t)$. First we calculate the partial derivatives involved,

- (i). $\frac{\partial Y}{\partial t} = 0$, since Y is not a function of time.
- (ii). $\frac{\partial Y}{\partial S} = \frac{1}{S}$, $\ln(S)$ with respect to S is $\frac{1}{S}$
- (iii). $\frac{\partial^2 Y}{\partial S^2} = \frac{-1}{S^2}$, the second derivative of $\ln(S)$

Let the partial derivatives (ii) be equation (19) and (iii) be equation (20). Plugging the values into Ito's lemma equation (17)

$$dY = (0 + \mu S \frac{1}{S} + \frac{1}{2} \sigma^2 S^2 (\frac{-1}{S^2})) dt + \sigma S \frac{1}{S} dW \quad (21)$$

From equation 21, we obtain;

$$dY = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW \quad (22)$$

Since $Y = \ln(S_t)$, we have;

$$d(\ln(S_t)) = (\mu - \frac{1}{2} \sigma^2) dt + \sigma dW \quad (23)$$

Integrating equation 23 with respect to t

$$\ln(S_t) - \ln(S_0) = (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t \quad (24)$$

Simplifying equation 24, we obtain;

$$\ln(\frac{S_t}{S_0}) = (\mu - \frac{1}{2} \sigma^2) t + \sigma W_t \quad (25)$$

Taking exponential of both sides from equation 25, we obtain;

$$S_t = S_0 \exp^{(\mu - \frac{1}{2} \sigma^2) t + \sigma W_t} \quad (26)$$

3.2 Derivation of Stock price following Geometric Brownian motion with jumps.

$$dS_t = S_t(\mu - \lambda k) dt + S_t \sigma dW_t + S_t dq_t, \quad (27)$$

Equation 27 can be solved using Ito's lemma given by;

$$dY = \left(\frac{\partial Y}{\partial t} + (\mu - \lambda t)S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2} \right) dt + \sigma S \frac{\partial Y}{\partial S} dW + S \frac{\partial Y}{\partial S} dq_t \quad (28)$$

From equation 28, we want to find dY where Y is defined as $\ln(S_t)$. First we obtain the partial derivatives involved.

- (i). $\frac{\partial Y}{\partial t} = 0$, since Y is not a function of time.
- (ii). $\frac{\partial Y}{\partial S} = \frac{1}{S}$, $\ln(S)$ with respect to S is $\frac{1}{S}$
- (iii). $\frac{\partial^2 Y}{\partial S^2} = \frac{-1}{S^2}$, the second derivative of $\ln(S)$

Plugging the values ito's lemma in equation 28

$$dY = \left(0 + (\mu - \lambda t)S \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \left(\frac{-1}{S^2} \right) \right) dt + \sigma S \frac{1}{S} dW + S \frac{1}{S} dq_t \quad (29)$$

Simplifying equation 29, we obtain;

$$dY = \left((\mu - \lambda t) - \frac{1}{2}\sigma^2 \right) dt + \sigma dW + dq_t \quad (30)$$

Since $Y = \ln(S_t)$, we have;

$$d(\ln(S_t)) = \left(\mu - \lambda t - \frac{1}{2}\sigma^2 \right) dt + \sigma dW + dq_t \quad (31)$$

Integrating equation 31 with respect to t

$$\ln(S_t) - \ln(S_0) = \left(\mu - \lambda t - \frac{1}{2}\sigma^2 \right) t - t_0 + \sigma W_t + q_t \quad (32)$$

Simplifying further equation 32, we obtain;

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \lambda t - \frac{1}{2}\sigma^2 \right) t - t_0 + \sigma W_t + q_t \quad (33)$$

Taking exponential of both sides from equation 33, we obtain;

$$S_t = S_0 \exp^{(\mu - \lambda t - \frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t} \quad (34)$$

3.3 Derivation of the price of a jump diffusion Geometric Brownian motion with the re-invested dividend and transaction cost.

Considering Yin Li *et al.* [5], Robert C. Merton [2] and Opondo [3], a jump diffusion Geometric Brownian motion with the dividend yielding and transaction cost equation is given by;

$$dS_t = S_t(\mu - \gamma - \tau - \lambda k)dt + S_t\sigma dW_t + S_t dq_t, \quad (35)$$

Where γ is the dividend yielding proportion and τ is the transaction cost proportion. By considering the reinvested dividend proportion and transaction cost, the resulting equation from 35 is given by;

$$dS_t = S_t(\mu + \gamma - \tau - \lambda k)dt + S_t\sigma dW_t + S_t dq_t, \quad (36)$$

Equation 36 can be solved using Ito's lemma given by;

$$dY = \left(\frac{\partial Y}{\partial t} + (\mu + \gamma - \tau - \lambda t)S \frac{\partial Y}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 Y}{\partial S^2}\right)dt + \sigma S \frac{\partial Y}{\partial S} dW + S \frac{\partial Y}{\partial S} dq_t \quad (37)$$

From equation 37, we want to find dY where Y is defined as $\ln(S_t)$. First we obtain the partial derivatives involved.

- (i). $\frac{\partial Y}{\partial t} = 0$, since Y is not a function of time.
- (ii). $\frac{\partial Y}{\partial S} = \frac{1}{S}$, $\ln(S)$ with respect to S is $\frac{1}{S}$
- (iii). $\frac{\partial^2 Y}{\partial S^2} = \frac{-1}{S^2}$, the second derivative of $\ln(S)$

Plugging the values ito's lemma in equation 37

$$dY = \left(0 + (\mu + \gamma - \tau - \lambda t)S \frac{1}{S} + \frac{1}{2}\sigma^2 S^2 \left(\frac{-1}{S^2}\right)\right)dt + \sigma S \frac{1}{S} dW + S \frac{1}{S} dq_t \quad (38)$$

Simplifying equation 38, we obtain;

$$dY = \left((\mu + \gamma - \tau - \lambda t) - \frac{1}{2}\sigma^2\right)dt + \sigma dW + dq_t \quad (39)$$

Since $Y = \ln(S_t)$, we have;

$$d(\ln(S_t)) = \left(\mu + \gamma - \tau - \lambda t - \frac{1}{2}\sigma^2\right)dt + \sigma dW + dq_t \quad (40)$$

Integrating equation 40 with respect to t

$$\ln(S_t) - \ln(S_0) = \left(\mu + \gamma - \tau - \lambda t - \frac{1}{2}\sigma^2\right)t - t_0 + \sigma W_t + q_t \quad (41)$$

Simplifying further equation 41, we obtain;

$$\ln\left(\frac{S_t}{S_0}\right) = (\mu + \gamma - \tau - \lambda t - \frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t \quad (42)$$

Taking exponential of both sides from equation 42, we obtain;

$$S_t = S_0 \exp^{(\mu + \gamma - \tau - \lambda t - \frac{1}{2}\sigma^2)t - t_0 + \sigma W_t + q_t} \quad (43)$$

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