

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(3): 82-92
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 10-01-2025
Accepted: 14-02-2025

Sher Singh Raikhola
Ph.D., Department of
Mathematics, Bhaktapur
Multiple Campus, Tribhuvan
University, Nepal

Real numbers (R) in fixed point theory and matrix spaces with applications

Sher Singh Raikhola

DOI: <https://www.doi.org/10.22271/maths.2025.v10.i3b.2009>

Abstract

This paper explores the applications of Fixed-point theory in metric spaces, focusing on the role of real numbers in matrix spaces and iterative methods. Using Banach's Contraction Principle, the study provides mathematical proofs to establish the existence and uniqueness of fixed points, while also analyzing the convergence and stability of iterative methods. In a complete metric space (X, d) , a contraction mapping $f: X \rightarrow X$ has a unique fixed point x^* such that $f(x^*) = x^*$ and $d(f(x), f(y)) \leq \alpha d(x, y)$, where $0 \leq \alpha < 1$.

The Intermediate Value Theorem ensures that for a continuous function f on $[a, b]$ with $f(a) \leq L \leq f(b)$, there is a point $c \in [a, b]$ such that $f(c) = L$. The Mean Value Theorem states that for a differentiable function f on (a, b) , there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. The Cauchy-Schwarz inequality for vectors u and v gives $|\langle u, v \rangle| \leq \|u\| \|v\|$. For the function $(x) = \frac{1}{2}(x+3)$, the fixed point is $x^* = 3$. In matrix space, the operator $(A) = \frac{1}{2}A$ has a fixed point $A = 0$. For $x = \cos(x)$, the successive approximations method converges to $x^* \approx 0.7391$, and the Newton-Raphson method achieves faster convergence to the same fixed point.

This research highlights the practical applications of fixed-point theory, supported by real numbers, as a foundation for accurate mathematical models. It enhances outcome prediction in optimization, stability analysis, and system modeling, emphasizing the critical role of real numbers in both theoretical and applied mathematics.

Keywords: Real numbers, fixed point theory, matrix spaces, numerical analysis, convergence

Introduction

Mathematical Foundation and Context

Real numbers form a fundamental structure in mathematics, underpinning a wide range of theories and applications. In the context of Fixed-point theory and matrix spaces, real numbers serve as the essential numerical framework that facilitates the rigorous study of mathematical phenomena. To fully appreciate their role, it is important to understand the mathematical properties of real numbers, how they influence fixed point theory, and their application in matrix spaces ^[20].

Fixed Point Theory: Mathematical Perspective

Fixed point theory is concerned with identifying points $x \in X$ in a space X such that for a given function $f: X \rightarrow X$, the equation $f(x) = x$ holds. In metric spaces, particularly those defined over real numbers, fixed point theorems like Banach's Contraction Principle provide necessary conditions for the existence and uniqueness of fixed points. The mathematical significance of fixed points lies in their stability properties, which are leveraged in solving differential equations, optimization problems, and even in iterative algorithms used in numerical analysis ^[6].

Corresponding Author:
Sher Singh Raikhola
Ph.D., Department of
Mathematics, Bhaktapur
Multiple Campus, Tribhuvan
University, Nepal

Matrix Spaces: Handling Multidimensional Data

Let $M_{n \times m}(R)$ denote the vector space of $n \times m$ matrices with real entries. Each element of $M_{n \times m}(R)$ is a matrix $A = [a_{ij}]$ where $a_{ij} \in R$ for $i = 1, \dots, n$ and $j = 1, \dots, m$. Matrix spaces $M_{n \times m}(R)$ provide a framework for the representation and manipulation of multidimensional data. These spaces are fundamental in linear algebra, functional analysis, and applied mathematics. The operations within matrix spaces, including matrix multiplication AB (where $A \in M_{n \times k}(R)$ and $B \in M_{k \times m}(R)$), eigenvalue computation of square matrices $A \in M_{n \times n}(R)$, and transformation analysis, are deeply connected to the properties of real numbers R [12].

We will discuss Banach fixed point theorem in metric spaces with complete normed spaces and related topics.

Banach Fixed Point Theorem in Metric Spaces

The Banach Fixed Point Theorem, also known as the Contraction Mapping Theorem, is a fundamental result in metric space theory and analysis, particularly in the context of complete normed spaces. It provides conditions under which a function, or mapping, has a unique fixed point. This theorem has wide-ranging applications in various fields, including numerical analysis, differential equations, and optimization [13].

Metric Space

Let X be a non-empty set. A metric on X is a function $d: X \times X \rightarrow \mathbb{R}$, which assigns a non-negative real number to each pair of points in X . This number represents the "distance" between the points, and the function d must satisfy the following properties for all $x, y, z \in X$:

Non-negativity: $d(x, y) \geq 0$.

This property ensures that the distance between any two points is always non-negative.

Identity of Indiscernible: $d(x, y) = 0$ if and only if $x = y$.

This property states that the distance between two distinct points is positive, and the distance between a point and itself is zero [14].

Symmetry: $d(x, y) = d(y, x)$.

This ensures that the distance from x to y is the same as the distance from y to x .

Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$ [18].

This property is essential in ensuring that the direct distance between two points is always less than or equal to the distance measured by taking a detour through a third point z .

When a set X is equipped with a metric d , the pair (X, d) is called a metric space. This structure allows for the rigorous discussion of concepts like convergence, continuity, and compactness.

In fixed point theory, real numbers play a crucial role, particularly in the context of metric spaces.

Convergence and Limit of a Sequence

In a metric space (X, d) , a sequence $\{x_n\}$ is said to be convergent if there exists an element $x \in X$ such that: $\lim_{n \rightarrow \infty} d(x_n, x) = 0$

Here, x is called the limit of the sequence $\{x_n\}$, and we write this as $\lim_{n \rightarrow \infty} x_n = x$. Intuitively, this means that as n increases, the elements of the sequence $\{x_n\}$ get arbitrarily close to x [20].

Complete Metric Space

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point within X .

A Cauchy sequence is a sequence $\{x_n\}$ in a metric space (X, d) such that for every $\epsilon > 0$, there exists a positive integer N such that for all $p, q > N$, the distance $d(x_p, x_q) < \epsilon$. This means that the elements of the sequence become arbitrarily close to each other as the sequence progresses [17].

The importance of completeness lies in the fact that it ensures the existence of limits within the space, which is a crucial property for various mathematical constructions and proofs.

Fixed Point

A fixed point of a mapping $f: X \rightarrow X$ is a point $x \in X$ such that $f(x) = x$. In other words, applying the function f to x returns x itself.

The Banach Fixed Point Theorem states that if (X, d) is a complete metric space and $f: X \rightarrow X$ is a contraction mapping (i.e., there exists a constant $c \in [0, 1)$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$), then f has a unique fixed point in X . Moreover, the fixed point can be found as the limit of the sequence defined by $x_{n+1} = f(x_n)$, starting from an arbitrary point $x_0 \in X$ [1].

Integrating Fixed Point Theory with Matrix Spaces

This research aims to bridge the mathematical theory of fixed points with the practical framework provided by matrix spaces, focusing on the role that real numbers play in this integration. Specifically, we investigate how real numbers contribute to defining and understanding fixed points within matrix spaces, examining key concepts such as:

Contraction Mappings in Matrix Spaces: The role of real numbers in defining contraction mappings, which are critical in establishing fixed points [2, 5].

Let (X, d) be a metric space, where X is a set of matrices and $d: X \times X \rightarrow \mathbb{R}$ is a metric. A mapping $T: X \rightarrow X$ is a contraction mapping if there exists a constant $c \in [0, 1)$ such that:

$$d(T(A), T(B)) \leq c \cdot d(A, B) \quad \forall A, B \in X$$

where c is a constant such that $0 \leq c < 1$.

Here, the role of real numbers c in defining the contraction is crucial in ensuring the existence and uniqueness of fixed points A^* in X such that $T(A^*) = A^*$ ^[18].

By focusing on these mathematical aspects, the research seeks to provide a comprehensive analysis of how real numbers facilitate the study and application of fixed-point theory within the context of matrix spaces. This integration not only enhances our understanding of these concepts but also offers practical tools for solving complex mathematical problems in various disciplines.

Statement of Problem

The intersection of fixed-point theory and matrix spaces, particularly through the framework of real numbers, introduces significant complexities. Investigating the existence and uniqueness of fixed points within matrix spaces demands an in-depth exploration of real numbers and their role in the behavior of iterative methods. This study tackles the issue of delineating the impact of real numbers on fixed-point theory within matrix spaces, examining how these characteristics can be applied in real-world scenarios. The core challenge lies in comprehending the influence of real numbers on fixed-point existence and uniqueness, with a focus on their application in iterative processes and mathematical modeling, thus addressing the intricate nature of fixed points in matrix spaces.

Objectives

The primary objectives of this research are:

- To analyze the role of real numbers in fixed point theory and matrix spaces.
- To establish conditions for the existence and uniqueness of fixed points in various matrix spaces.
- To explore the Applications of real numbers in fixed point theory.
- To apply the findings in mathematical modeling and numerical analysis.

Methodology

The methodology for this research combined theoretical analysis with practical applications to investigate fixed point theory within metric spaces. Theoretical foundations were established by defining the basic properties of metric spaces and exploring contraction mappings to determine conditions for the existence of unique fixed points. Key theoretical results were incorporated, such as the Archimedean property, which ensures that for any positive number and any real number, there is a natural number such that the product of these numbers exceeds the real number; the Bolzano-Weierstrass theorem, which guarantees that every bounded sequence in the real numbers has a convergent subsequence; the Intermediate Value Theorem, which asserts that for a continuous function on a closed interval, there is a point where the function attains any value between its values at the endpoints of the interval; the Mean Value Theorem, which provides insights into the behavior and rate of change of functions; and the Cauchy-Schwarz Inequality, used to evaluate the stability and convergence of iterative processes.

Practical examples were then utilized to illustrate these theoretical concepts. For example, the fixed point for a specific function was computed, resulting in a particular value, and the fixed point of an operator in the matrix space was found to be zero. The research also applied iterative methods, such as successive approximations and the Newton-Raphson method, to solve equations. Successive approximations demonstrated convergence to a specific value, while the Newton-Raphson method achieved faster convergence to the same fixed point. The convergence and stability of these iterative methods were analyzed through both theoretical results and practical evidence, providing a comprehensive understanding of fixed point theory and its real-world applications.

Results

In this section, we present the main findings of our research, structured around the objectives.

Role of Real Numbers in Fixed Point Theory

Real numbers play a critical role in defining the metric spaces where fixed points are analyzed. We provide a detailed examination of the conditions under which real numbers ensure the existence and uniqueness of fixed points, particularly focusing on contraction mappings in matrix spaces ^[20, 9].

Completeness Axiom (Completeness Property)

Every non-empty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum $\sup s \in \mathbb{R}$ ^[17].

Banach Fixed Point Theorem: In a complete metric space (X, d) , if $f: X \rightarrow X$ is a contraction, then f has a unique fixed point $x^* \in X$, where $x^* = f(x^*)$.

Mathematical Formulation: There exists a unique $x^* \in X$ such that $f(x^*) = x^*$ and

$$d(f(x), f(y)) \leq \alpha d(x, y) \text{ with } 0 \leq \alpha < 1.$$

Archimedean Property

For $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$ ^[20].

Example 1: Fixed Point and Real Numbers in a Bounded Setting

Problem: Verify that a function $f(x) = x^2$ has fixed points within finite bounds when considering x greater than a certain positive value.

Given: Function: $f(x) = x^2, x > 0$ We need to ensure that for a natural number n , if $n \cdot x > 1$, then the values are finite and computable

Solution: Fixed Point Definition: A fixed point of a function (x) is a value x^* such that $f(x^*) = x^*$.

For $f(x) = x^2$, we find fixed points by solving: $x^2 = x$

Solving the Fixed-Point Equation: Rearrange the equation: $x^2 - x = 0$

Factorize: $x(x - 1) = 0$, So the solutions are: $x = 0$ or $x = 1$

The fixed points are $x = 0$ and $x = 1$. However, we are given $x > 0$, so the relevant fixed point is $x = 1$.

Bounded Values Condition: We need to ensure that $n \cdot x > 1$ for natural numbers n .

For $x = 1$:

$n \cdot 1 > 1 \Rightarrow n > 1$ So, if n is any natural number greater than 1, the condition $n \cdot x > 1$ is satisfied for $x = 1$.

Verification of Finite Bounds: To ensure that no "infinitely large" values distort the calculations, we verify that x is bounded. In this case:

$x = 1$ is finite and within the bounds set by $n \cdot x > 1$ for $n > 1$.

Therefore, the fixed point $x = 1$ is finite and computable.

The function $f(x) = x^2$ has a fixed point $x = 1$ when $x > 0$. The condition $n \cdot x > 1$ for natural numbers $n > 1$ ensures that x does not become infinitely large. Thus, the fixed point is within finite bounds and is valid within the given constraints.

Bolzano-Weierstrass Theorem

Every bounded sequence $\{x_n\} \subseteq R$ has a convergent subsequence $\{x_{n_k}\}$ with

$$x_{n_k} \rightarrow L \in R^{[9]}.$$

This theorem implies that if $\{x_n\}$ is a sequence generated by an iterative method in a complete metric space, there is a convergent subsequence $\{x_{n_k}\}$ that converges to a fixed point x^* . Mathematically, this means

For a sequence $\{x_n\}$ in a complete metric space, there exists x^* such that

$$\lim_{n \rightarrow \infty} d(x_{n_k}, x^*) = 0. \text{ where } d \text{ is the metric on the space.}$$

Example 2: Consider the iterative method $x_{n+1} = \frac{x_n + 1}{2}$ with $x_0 = 0$. The sequence $\{x_n\}$ generated by this method is bounded and converges to the fixed point $x^* = 1$, demonstrating the Bolzano-Weierstrass Theorem's relevance in confirming the convergence to x^* .

Intermediate Value Theorem

If $f: [a, b] \rightarrow R$ is continuous and $L \in ((a), (b))$, then there exists $c \in [a, b]$ such that $f(c) = L$ [4].

For a continuous function f , if $f(a)$ and $f(b)$ are on opposite sides of L , then there exists $c \in [a, b]$ such that $f(c) = L$. Specifically, if $f(a) < 0$ and $f(b) > 0$, then there exists $c \in [a, b]$ such that $f(c) = 0$.

Example 3: Let $f(x) = x^2 - 2$ on the interval $[1, 2]$. Since $f(1) = -1$ and $f(2) = 2$, and $0 \in [-1, 2]$, by the Intermediate Value Theorem, there exists $c \in [1, 2]$ such that $f(c) = 0$. Here, $c = \sqrt{2}$ is a fixed point where $f(c) = 0$.

Mean Value Theorem

If $f: [a, b] \rightarrow R$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists

$$c \in (a, b) \text{ such that } f'(c) = \frac{f(b) - f(a)}{b - a} \text{ [15].}$$

Rate of Change Analysis provides the rate of change of f to optimize and analyze fixed point functions.

Example 4

Consider the function $f(x) = \sin(x)$ and we are analyzing it over the interval $[0, \pi]$. We want to find the fixed point and then use the rate of change to analyze the function around this fixed point.

Step 1: Find a Fixed Point

A fixed point x^* satisfies the equation $f(x^*) = x^*$.

Set up the equation: $\sin(x^*) = x^*$

This equation cannot be solved algebraically, so we use numerical methods to approximate the fixed point.

Using numerical methods (such as Newton's method), we find that the fixed point in the interval $[0, \pi]$ is approximately $x^* \approx 0.88$.

Step 2: Rate of Change Analysis

Let's use the fixed point $x^* \approx 0.88$ and analyze the rate of change around this point.

Compute the derivative $f'(x)$: $f'(x) = \cos(x)$

Choose a point x in the interval around the fixed point. For simplicity, let's use $x = 0.9$ (which is close to x^*).

Apply the mathematical formulation

$$f'(c) = \frac{f(x^*) - f(x)}{x^* - x}$$

Calculate $f(x)$ and $f(x^*)$:

$$f(x^*) = \sin(0.88) \approx 0.77$$

$$f(x) = \sin(0.9) \approx 0.78$$

Compute the rate of change

$$f'(c) = \frac{0.77 - 0.78}{0.88 - 0.9}$$

$$f'(c) = \frac{-0.01}{-0.02} = 0.5$$

Therefore, $f'(c) = 0.5$.

The rate of change $f'(c) = 0.5$ at the fixed point $x^* \approx 0.88$ indicates how rapidly the function $f(x) = \sin(x)$ is changing around this fixed point. This information can be useful in optimization problems, particularly in understanding the behavior of f near the fixed point and refining x^* for optimal solutions. For example, if the rate of change is small, the function is changing slowly around the fixed point, which might suggest that x^* is a stable or nearly optimal point.

Cauchy-Schwarz Inequality

For vectors u, v in an inner product space, $|\langle u, v \rangle| \leq \|u\| \|v\|$ ^[4].

In an inner product space, $|\langle u, v \rangle| \leq \|u\| \|v\|$ ensures stability in iterative algorithms ^[21].

Example 5

Consider the iterative method for finding the fixed point of $f(x) = \frac{1}{2}(x + \frac{2}{x})$, where x is the variable and the fixed point x^* is $\sqrt{2}$. Using the Cauchy-Schwarz inequality, we can analyze the stability and convergence of this method by bounding the inner product $\langle x_n, x_{n+1} \rangle$ with the norms of x_n and x_{n+1} , ensuring that the iterates remain close to the fixed point $\sqrt{2}$.

Fixed Point Definition

Fixed point theory is concerned with identifying points $x \in X$ in a space X such that for a given function $f: X \rightarrow X$, the equation $f(x) = x$ holds ^[3].

Example 6

Consider the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by: $f(x) = \frac{1}{2}(x + 3)$.

We want to find a fixed point x^* such that $f(x^*) = x^*$.

Step 1: Set Up the Fixed-Point Equation

The Fixed-point equation for this function is: $f(x) = x$.

Substituting the definition of $f(x)$: $\frac{1}{2}(x + 3) = x$.

Step 2: Solve for the Fixed Point

To solve for x , first eliminate the fraction by multiplying both sides by 2: $x + 3 = 2x$.

Now, rearrange the equation to isolate x : $3 = 2x - x$, $3 = x$.

Thus, the fixed point is $x^* = 3$.

Step 3: Verify the Fixed Point $x^* = 3$.

To verify that is indeed a fixed point, substitute $x=3$ back into the function: $f(3) = \frac{1}{2}(3 + 3) = \frac{1}{2} \times 6 = 3$

Since $f(3) = 3$, the point $x^* = 3$ satisfies the fixed-point equation $f(x) = x$.

Interpretation

In this example, the function $\frac{1}{2}(x + 3) = x$ has a fixed point at $x^* = 3$, which is the solution to the equation $f(x) = x$. The fixed point $x^* = 3$ is a point in the space $\mathbb{R}$ where the function f maps the point to itself, meaning that $f(x^*) = x^*$.

Metric Spaces and Fixed Points

A metric space (X, d) consists of a set X and a distance function $d: X \times X \rightarrow \mathbb{R}$ satisfying:

Non-negativity: $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in X$.

Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A fixed point x^* of a function $T: X \rightarrow X$ satisfies: $T(x^*) = x^*$ [3], [1].

This means that applying the function T to x^* leaves it unchanged.

Example 7

Consider $X = \mathbb{R}$, the set of real numbers, with the standard metric (distance function) $d(x, y) = |x - y|$. Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined as: $T(x) = \frac{x}{2}$

We want to find the fixed point of this function. A fixed point x^* satisfies: $T(x^*) = x^*$

Substitute the function $T(x)$: $\frac{x^*}{2} = x^*$, $2x^* - x^* = 0$, $x^*(2 - 1) = 0$, i.e. $x^* = 0$. Thus, $x^* = 0$ is the fixed point of the function $(x) = \frac{x}{2}$

Verification in Metric Space

In the metric space $(\mathbb{R}, d)$, with $d(x, y) = |x - y|$, let's verify if T satisfies the conditions for a contraction mapping (a common context for fixed point theorems).

A mapping $T: X \rightarrow X$ is called a contraction if there exists a constant $c \in [0, 1)$ such that:

$$d(T(x), T(y)) \leq c \cdot d(x, y)$$

For $T(x) = \frac{x}{2}$, let's compute the distance:

$$d(T(x), T(y)) = \left| \frac{x}{2} - \frac{y}{2} \right| = \frac{1}{2} |x - y|$$

Thus, T is a contraction with constant $c = \frac{1}{2}$, and by the Banach Fixed Point Theorem, the function $T(x) = \frac{x}{2}$ has a unique fixed point, which we already found as $x^* = 0$.

This example demonstrates the use of real numbers in fixed point theory within a metric space. The function $T(x) = \frac{x}{2}$ has a unique fixed point at $x^* = 0$, and the contraction property ensures its existence and uniqueness.

Contraction Mappings

A mapping $T: X \rightarrow X$ is called a contraction mapping if there exists a constant $\alpha \in [0, 1)$ such that:

$d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in X$. The constant α and the metric d are defined using real numbers, which determine the extent to which T contracts distances [5, 12].

Let's consider a mathematical example of a Contraction Mapping in a real number context using fixed point theory:

Example 8

Let $X = \mathbb{R}$ (the set of real numbers) and let the metric $d(x, y)$ be the usual absolute difference:

$$d(x, y) = |x - y|$$

Now, define a function

$T: \mathbb{R} \rightarrow \mathbb{R}$ as:

$$T(x) = \frac{1}{2}x + 1$$

We want to show that T is a contraction mapping, and then use the Banach Fixed Point Theorem to find the fixed point of T .

Step 1: Show that T is a contraction mapping.

We need to find a constant $\alpha \in [0, 1)$ such that for all $x, y \in \mathbb{R}$, the following inequality holds:

$$d(T(x), T(y)) = |T(x) - T(y)| \leq \alpha |x - y|$$

Substitute the definition of $T(x)$ and $T(y)$ into the inequality:

$$|T(x) - T(y)| = \left| \left(\frac{1}{2}x + 1 \right) - \left(\frac{1}{2}y + 1 \right) \right| = \left| \frac{1}{2}(x - y) \right| = \frac{1}{2} |x - y|$$

Thus, the function T satisfies the inequality with $\alpha = \frac{1}{2}$, which is clearly in the interval $[0, 1)$. Therefore, T is a contraction mapping with contraction constant $\alpha = \frac{1}{2}$.

Step 2: Apply the Banach Fixed Point Theorem.

The Banach Fixed Point Theorem assures that if T is a contraction mapping on a complete metric space X , then T has a unique fixed point. A fixed point x^* satisfies the equation:

$$T(x^*) = x^*$$

Substitute the definition of $T(x)$: $\frac{1}{2}x^* + 1 = x^*$

$$\text{Solve for } x^*: 1 = x^* - \frac{1}{2}x^* = \frac{1}{2}x^*, x^* = 2$$

The function $T(x) = \frac{1}{2}x + 1$ is a contraction mapping on $\mathbb{R}$ with contraction constant $\alpha = \frac{1}{2}$, and it has a unique fixed point at $x^* = 2$.

Banach Fixed Point Theorem

The Banach Fixed Point Theorem states that if (X, d) is a complete metric space and $T: X \rightarrow X$ is a contraction mapping, then T has a unique fixed point in X [1, 13].

Proof of Existence and Uniqueness Using Real Numbers

Existence

Cauchy Sequence: Let (X, d) be a complete metric space and T be a contraction mapping with constant α . Consider an arbitrary initial point $x_0 \in X$ and define the sequence $\{x_n\}$ by:

$$x_{n+1} = T(x_n)$$

Convergence: For $m > n$, we have:

$$d(x_{m+1}, x_n) = d(T(x_m), T(x_n)) \leq \alpha^m d(x_0, x_n).$$

Since $\alpha < 1$, $\alpha^m \rightarrow 0$ as $m \rightarrow \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence because:

$$d(x_{m+1}, x_n) \leq \alpha^m d(x_0, x_n).$$

Fixed Point Verification: To show x^* is a fixed point, observe:

$$x_{n+1} = T(x_n) \text{ Taking the limit as } n \rightarrow \infty: x^* = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} T(x_n)$$

Since T is continuous, we have: $x^* = T(x^*)$. Hence, x^* is a fixed point.

Uniqueness

Proof of Uniqueness: Suppose x^* and y^* are both fixed points. such that $T(x^*) = x^*$ and $T(y^*) = y^*$. Then

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq \alpha \cdot d(x^*, y^*).$$

Since $\alpha < 1$, It follows that: $(x^*, y^*) = \alpha \cdot d(x^*, y^*)$.

This implies: $(x^*, y^*) = 0, \Rightarrow x^* = y^*$. Hence, the fixed point is unique.

Example 9

Example: Matrix Space and Contraction Mapping

Consider the matrix space $M_{2 \times 2}(\mathbb{R})$ with the Frobenius norm $\|A\|_F = \sqrt{\text{trace}(A^T A)}$.

Define the mapping $T: M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ by:

$$T(A) = \frac{1}{2}A.$$

Verify Contraction Mapping

Contraction Property: For matrices $A, B \in M_{2 \times 2}(\mathbb{R})$,

$$\|T(A) - T(B)\|_F = \left\| \frac{1}{2}A - \frac{1}{2}B \right\|_F = \frac{1}{2} \|A - B\|_F.$$

Thus, T is a contraction mapping with constant $\alpha = \frac{1}{2}$.

Apply Banach Fixed Point Theorem

Complete Metric Space: $M_{2 \times 2}(R)$ with the Frobenius norm is a complete metric space.

Existence: By the Banach Fixed Point Theorem, T has a unique fixed point.

Find the Fixed Point

Solve for Fixed Point: Set $T(A) = A$:

$$\frac{1}{2}A = A \Rightarrow A = 0. \text{ Hence, } A=0 \text{ is the unique fixed point of } T \text{ in } M_{2 \times 2}(R) \text{ [25].}$$

Hence, Real numbers are essential in defining the metric space and contraction mappings used in fixed point theory. The Banach Fixed Point Theorem demonstrates that, with real-valued metrics and contraction constants, matrix spaces guarantee the existence and uniqueness of fixed points.

Mathematical Significance of Fixed Points

The mathematical significance of fixed points lies in their stability properties, which are leveraged in solving differential equations, optimization problems, and even in iterative algorithms used in numerical analysis [7, 14].

Proof (Stability in Dynamical Systems)

Stability Concept: Consider a differential equation $dx/dt = f(x)$ with equilibrium points x^* such that $f(x^*) = 0$. These points correspond to fixed points of the function $f(x) = x + (x)$.

Linear Stability Analysis: Linearize the system around the fixed point x^* :

$$dx/dt \approx f'(x^*) \cdot (x - x^*).$$

The stability of x^* is determined by the sign of $f'(x^*)$. If $f'(x^*) < 0$, the fixed point is stable; if $f'(x^*) > 0$, it is unstable.

Proof (Optimization Problems)

Optimization and Fixed Points: Consider the gradient descent method, where the fixed point x^* of the gradient function $\nabla f(x) = 0$ represents a local minimum.

Iterative Algorithm: Define the update rule $x_{n+1} = x_n - \gamma \nabla f(x_n)$. The fixed point x^* satisfies:

$x^* = x^* - \gamma \nabla f(x^*)$, implying $\nabla f(x^*) = 0$, i.e., x^* is a critical point, often corresponding to a minimum.

Proof (Numerical Analysis and Iterative Methods):

Convergence to Fixed Points: Numerical methods like the Newton-Raphson method aim to find solutions to $f(x)=0$. The sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$(x) = x - \frac{f(x)}{f'(x)}, \text{ under appropriate conditions on } f.$$

Fixed point theory, via Banach's Contraction Principle, proves the existence and uniqueness of solutions in metric spaces. It is essential in ensuring stability, finding optimal solutions, and achieving convergence in various mathematical fields.

Example 10

Let's solve the example application using numerical analysis with the Newton-Raphson method and the successive approximations method for finding the fixed point of the equation

$$x = \cos(x).$$

Step 1: Analyze Convergence

First, we analyze the convergence behavior by checking the derivative of $\cos(x)$.

The function $f(x) = \cos(x)$ has the derivative: $f'(x) = -\sin(x)$

For convergence in fixed point iterations, we require: $|f'(x)| = |\sin(x)| < 1$

Since $|\sin(x)| \leq 1$ for all x , the condition is met. However, we must ensure $|\sin(x)|$ is significantly less than 1 for rapid convergence.

Step 2: Successive Approximations Method

We use the iteration: $x_{n+1} = \cos(x_n)$

Let's choose an initial guess $x_0 = 1.0$ and iterate:

$$x_1 = \cos(1.0) \approx 0.5403$$

$$x_2 = \cos(0.5403) \approx 0.8576$$

$$x_3 = \cos(0.8576) \approx 0.6543$$

$$\begin{aligned}
x_4 &= \cos(0.6543) \approx 0.7937 \\
x_5 &= \cos(0.7937) \approx 0.7074 \\
x_6 &= \cos(0.7074) \approx 0.7645 \\
x_7 &= \cos(0.7645) \approx 0.7221 \\
x_8 &= \cos(0.7221) \approx 0.7504 \\
x_9 &= \cos(0.7504) \approx 0.7314 \\
x_{10} &= \cos(0.7314) \approx 0.7442
\end{aligned}$$

Notice that the values are converging. The fixed point is approximately $x^* \approx 0.7391$ [7].

Step 3: Newton-Raphson Method

Now, let's use the Newton-Raphson method. For the equation

$$x = \cos(x), \text{ rewrite it as: } g(x) = x - \cos(x) = 0$$

The derivative $g'(x)$ is: $g'(x) = 1 + \sin(x)$

The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

Starting with $x_0 = 1.0$:

$$x_1 = 1.0 - \frac{1.0 - \cos(1.0)}{1 + \sin(1.0)} = 1.0 - \frac{1.0 - 0.5403}{1 + 0.8415} \approx 1.0 - \frac{0.4597}{1.8415} \approx 1.0 - 0.2497 \approx 0.7503$$

$$x_2 = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1 + \sin(0.7503)} = 0.7503 - \frac{0.7503 - 0.7317}{1 + 0.6816} \approx 0.7503 - \frac{0.0186}{1.6816} \approx 0.7503 - 0.0111 \approx 0.7392$$

$$x_3 = 0.7392 - \frac{0.7392 - \cos(0.7392)}{1 + \sin(0.7392)} = 0.7392 - \frac{0.7392 - 0.7391}{1 + 0.6741} \approx 0.7392 - \frac{0.0001}{1.6741} \approx 0.7392 - 0.000059733 \approx 0.739140267 \approx 0.7391$$

The iteration results

$$\begin{aligned}
x_1 &\approx 0.7503 \\
x_2 &\approx 0.7392 \\
x_3 &\approx 0.7391 \text{ [19, 23].}
\end{aligned}$$

The solution converges to approximately $x \approx 0.7391$, which is a fixed point where $x = \cos(x)$

Here, the method converges rapidly to the fixed point $x^* \approx 0.7391$

Successive Approximations: This method shows gradual convergence, taking about 10 iterations to get close to the fixed point.

Newton-Raphson Method: This method converges faster, requiring only about 3 iterations.

The Newton-Raphson method is more efficient for this problem, given the faster convergence, but it requires the computation of the derivative. The successive approximations method, though slower, is simpler and does not require the derivative. Both methods ultimately lead to the same fixed point, $x^* \approx 0.7391$.

Applications of Fixed-Point Theorem

The Banach Fixed Point Theorem, which relies on real-valued metrics and contraction constants, guarantees the existence and uniqueness of fixed points in complete metric spaces. This theorem has broad applications across various fields:

Economics: General equilibrium models leverage fixed point theory to ensure that economic equilibria converge to a stable state, facilitating accurate predictions and analyses of market behaviors [22].

Engineering: In control systems, the stability analysis often involves matrix equations where real numbers define metric spaces and contraction mappings. This ensures that the control systems are robust and stable [16, 24].

Computer Science: Iterative algorithms frequently use real numbers to ensure the efficiency and accuracy of fixed point computations. This includes algorithms for optimization and numerical solutions where fixed points ensure convergence [10].

Physics: In dynamic systems analysis, fixed points represent stable states of systems. Real numbers provide the necessary framework for analyzing and predicting the behavior of these systems [11].

Discussion

Completeness Axiom

Our result showed that every non-empty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum $\sup S \in \mathbb{R}$. This is consistent with the Banach Fixed Point Theorem, which states that in a complete metric space (X, d) , if $T: X \rightarrow X$ is a contraction mapping, then T has a unique fixed point $x^* \in X$ such that $T(x^*) = x^*$ and $d(T(x), T(y)) \leq \alpha d(x, y)$ where $0 \leq \alpha < 1$. This finding corroborates the

results by Mannan *et al.*,^[13] and Agarwal *et al.*,^[2], confirming that completeness ensures the existence and uniqueness of fixed points in such metric spaces.

Archimedean Property

Our result showed that for any $x, y \in R$ with $x > 0$, there exists $n \in N$ such that $nx > y$. This prevents distortion from infinitely large values and is consistent with Majid *et al.*,^[12], who demonstrated that the Archimedean property ensures the finiteness and computability of fixed points.

Bolzano-Weierstrass Theorem

Our result confirmed that every bounded sequence $\{x_n\} \subseteq R$ has a convergent subsequence $\{x_{n_k}\}$ converging to some limit $L \in R$. This supports the convergence of iterative methods, ensuring that if $\{x_n\}$ is generated in a complete metric space, there exists a fixed point

x^* such that $\lim_{n \rightarrow \infty} d(x_{n_k}, x^*) = 0$. This aligns with findings by Howie^[9], who demonstrated that this theorem guarantees fixed points through convergent subsequences.

Intermediate Value Theorem

Our result indicated that if f is continuous on $[a, b]$ and L is a number between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = L$. This theorem supports the existence of fixed points for continuous functions within specified intervals. Buchanan^[7], similarly confirmed this, highlighting its role in ensuring the existence of fixed points in continuous functions.

Mean Value Theorem

Our findings showed that if f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. This theorem provides insight into the rate of change of f and is used to optimize and analyze fixed point functions. This is consistent with results by Agarwal *et al.*,^[1] and Yadav^[25], who applied this theorem for stability analysis and optimization in fixed point problems.

Cauchy-Schwarz Inequality

Our result confirmed that for any vectors u and v in an inner product space, $|\langle u, v \rangle| \leq \|u\| \|v\|$. ensures bounds and stability in fixed point iterations. This aligns with Shakhno^[21], who used this inequality to ensure boundedness and stability in iterative algorithms

Numerical Analysis

In solving $x = \cos x$, iterative methods such as Successive Approximations and Newton-Raphson are employed. Both methods converge to the fixed point $x^* \approx 0.7391$. However, the Newton-Raphson Method demonstrates superior efficiency, converging in approximately 3 iterations compared to the 10 iterations required by Successive Approximations. This highlights Newton-Raphson's practical effectiveness for faster convergence. The consistency in results between these methods underscores their reliability and the crucial role of real numbers in achieving precise solutions. Our findings align with Mannan *et al.*,^[13], who demonstrated the effectiveness of fixed point methods in various practical applications, and Agarwal *et al.*,^[1], who confirmed the utility of iterative methods like Newton-Raphson in optimizing computational efficiency. This supports the theoretical underpinnings of fixed point theory and its practical applications.

Conclusion

This research highlights the critical role of real numbers in fixed point theory and matrix spaces. It explores the foundational concepts of these mathematical areas, demonstrating how real numbers affect the existence, uniqueness, and convergence of fixed points in diverse contexts. The study examines the implications of real numbers for stability and solutions, offering practical methods for addressing complex problems across different domains. Through theoretical analysis and practical applications, the research underscores how real numbers enhance the understanding and effectiveness of fixed point theory in mathematical modeling and numerical techniques. The findings are particularly relevant for solving challenges in fields such as engineering and economics, illustrating the profound impact of real numbers on both theoretical and applied mathematics.

Acknowledgment

We sincerely thank previous researchers for their pioneering work on the application of real numbers in fixed-point theory and matrix spaces. Their contributions have significantly advanced our understanding and provided valuable insights that have paved the way for further research in this field.

Conflict of Interest

The authors declare no conflict of interest related to this research.

References

1. Agarwal P, Jleli M, editors. Fixed point theory in metric spaces: Recent advances and applications [Internet]. Springer; 2018 [cited YYYY MMM DD]. Available from: <https://doi.org/10.1007/978-981-13-2913-5>
2. Agarwal P, Jleli M, Samet B. Banach Contraction Principle and Applications. 2018.
3. Ansari QH, Sahu DR. Fixed point theory in metric spaces. Cambridge University Press; 2023. Published online 15 July.
4. Apostol TM. Mathematical Analysis. Addison-Wesley; c1974.

5. Bailey DF. Some theorems on contractive mappings. *Journal of the London Mathematical Society*. 1966;1(1):101-106.
6. Belluce LP, Kirk WA. Some fixed point theorems in metric and Banach spaces. Cambridge University Press; 2018. Published online 20 November.
7. Buchanan JR. Fixed-Point Iteration. MATH 375 Numerical Analysis. Department of Mathematics, Spring 2022.
8. Deepak, Meenakshi. Metric space and its applications. *Journal of University of Shanghai for Science and Technology*. 2021;23(7):957-961.
9. Howie JM. Real Analysis. Springer Undergraduate Mathematics Series. Springer; c2001. Available from: <https://doi.org/10.1007/978-1-4471-0341-7>
10. Knuth DE. The Art of Computer Programming, Volume 1: Fundamental Algorithms. 3rd ed. Addison-Wesley; c1997.
11. Landau LD, Lifshitz EM. Statistical Physics, Part 1. 3rd ed. Pergamon Press; c1980.
12. Majid NA, AL Jumaili A, Lee SK. Some applications of fixed point results for monotone multivalued and integral type contractive mappings. Research. Published 17 July 2023.
13. Mannan MA, Rahman MR, Akter H, Nahar N, Mondal S. A study of Banach fixed point theorem and its applications. *American Journal of Computational Mathematics*. 2021;11(2):157-174. Available from: <https://doi.org/10.4236/ajcm.2021.112011>
14. Martynyuk AA. Stability of dynamical systems in metric space. *Nonlinear Dynamics and Systems Theory*. 2005;5(2):157-167.
15. Matkowski J. A mean-value theorem and its applications. *Journal of Mathematical Analysis and Applications*. 2011;373(1):227-234. Available from: <https://doi.org/10.1016/j.jmaa.2010.07.031>
16. Ogata K. Modern Control Engineering. 5th ed. Prentice Hall; c2010.
17. Okelo NB. Properties of complete metric spaces and their applications to computer security, information and communication technology. 2014;1:23-28.
18. Palais RS. A simple proof of the Banach contraction principle. *Journal of Fixed Point Theory and Applications*. 2007;2(2):221-223. Available from: <https://doi.org/10.1007/s11784-007-0041-6>
19. Pho KH. Improvements of the Newton–Raphson method. *Journal of Computational and Applied Mathematics*. 2022;408:114106. Available from: <https://doi.org/10.1016/j.cam.2022.114106>
20. Rudin W. Principles of Mathematical Analysis. McGraw-Hill; c1976.
21. Shakhno SM. An iterative algorithm with superquadratic convergence for solving nonlinear operator equations. *Journal of Computational and Applied Mathematics*. 2009;231(1):222-235. Available from: <https://doi.org/10.1016/j.cam.2008.08.014>
22. Skare M, Stjepanovic S. How important are general equilibrium models for small open economies – the case of Croatia. *Technological and Economic Development of Economy*. 2013;19(2):331-349.
23. Triana JG. On the Newton-Raphson method and its modifications. *CIENCIA ENDESARROLLO*. 2023;14(2):75-80. Available from: <https://doi.org/10.19053/01217488.v14.n2.2023.15157>
24. Xue Y, Han J, Tu Z, Chen X. Stability analysis and design of cooperative control for linear delta operator system. *AIMS Mathematics*. 2023;8(6):12671-12693. Available from: <https://doi.org/10.3934/math.2023637>
25. Yadav D, Tiwari SK. Expansive type rational contraction in metric space and common fixed point theorems. *Mikailalsys Journal of Mathematics and Statistics*. 2023, 1(1)