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Resonant wave interactions

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Abstract

Interaction between non-linearity and dispersion is explained by using model equations for resonant wave interactions of geometrical acoustics. Lax-Friedrichs conservative difference scheme is modified for resonant wave interactions and complete elimination of shock wave solutions is demonstrated for a test case.

Keywords: Non-linearity, dispersion, resonant wave interactions

Introduction

Non-linear wave motion in fluids is classified broadly into dispersive and non-dispersive wave motions. Dispersive motions are a group of phenomena where wave speed varies with wavelength and also with the direction of propagation. In dispersive wave motion velocity takes different values for different sinusoidal components of a general disturbance. It is known in the literature that non-linear analysis of non-dispersive systems is almost of the same order of difficulty as the linear analysis of dispersive systems, with magnitude of the signal very small to affect a signal velocity that varies with signal wavelength. Gravity waves (Lighthill, 1978) are main examples of dispersive waves. Gravity waves are formed in fluids driven in order to find balance between a fluid's inertia and its tendency, under gravity, to return to a state of equilibrium with heavier fluid underlying lighter. It is known in the literature that the stratification of density within parts of the atmosphere is such that internal gravity waves form. Problems involving high frequency waves arise in many applications. They arise in particular in light waves propagating in dispersive medium, sound waves propagating through atmosphere and water waves propagating over gently sloping ocean bed.

John Hunter has derived a method for finding small amplitude high frequency solutions to systems of quasilinear partial differential equations of hyperbolic type. The solution generally is sum of the superposition of small amplitude high frequency waves with slowly varying mean solution. Each high frequency wave displays non-linear distortions of the wave profile and shock waves may form. Formation of shock waves is important feature of systems of quasilinear partial differential equations of hyperbolic type.

In this article we propose to study interaction between non-linearity and dispersion to prevent breakdown in the small amplitude high frequency waves which are found by the method of Hunter and later which incorporated resonance between different components. Different high frequency waves do not interact if the frequencies and wave numbers of two waves are not linearly related to those of the third wave. Multiple scale techniques are used to carry out the analysis. However, if frequencies and wave numbers of two waves are linearly related to those of third wave then resonant interaction takes place and wave motion ceases from being the local one and becomes non-local.

Basic Equations

In the case of equations of compressible fluid flows in one or several space dimensions, the equation for resonant wave interactions assume the canonical form (Andrew Majda, 1988):

$$u_t + \left(\frac{1}{2}u^2\right)_x + \int_0^1 K(x+y)v(y,t) dy = 0$$

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$$v_t - \left(\frac{1}{2}v^2\right)_x - \int_0^1 K(x+y)u(y,t) dy = 0$$

Here $u = u(x, t)$, $v = v(x, t)$ are periodic functions with period one and mean zero. The kernel $K = K(x)$ is a given function with period one and mean zero. Initial conditions are also specified for the equations as $u(x, 0) = u_0(x)$, $v(x, 0) = v_0(x)$.

$$u_t + \left(\frac{1}{2}u^2\right)_x + \int_0^1 K(x-y)v(y,t) dy = 0$$

$$v_t + \left(\frac{1}{2}v^2\right)_x + \int_0^1 K(-x+y)u(y,t) dy = 0$$

The above system has advantage of being invariant under translation. The above system is obtained from the former one by the change of variables from x to $-x$. The above system consists of two parts. First part is Burger's equation which is non-linear and exhibit shock wave solutions, the second part is the convolution integral which introduces non-locality and dispersion. In this article we will numerically investigate how non-linearity interacts with dispersion to prevent breakdown in the flow that is prevention of shock wave solutions. Shock waves caused because of convection terms will have opposing effects. The dispersion exchanges energy between u and v has cumulative effect of non-linearity working against itself as the decrease in the amplitude takes place and eventually eliminates the shock formation completely.

Numerical Method

The main difficulties whose care must be taken in the

development of numerical method for the non-linear problems is that of convergence, stability and consistency. The method should converge to some function as mesh size is made very small and it should be a solution to the problem. The issue with non-linear problems is that methods developed are often non-linearly unstable. That is method may appear to be stable on linear systems, but it will turn out to be unstable on non-linear problems. This problem of stability is solved by employing conservative methods and Lax-Friedrich's difference scheme is such a method which is proved to be stable for Burger's equations in the literature. We have modified Lax-Friedrichs scheme (LeVeque, 1992) by adding convolution term to it. Question of convergence, stability and consistency for this modified version of Lax-Friedrichs scheme will be discussed somewhere else. In this article our objective is to demonstrate exchange of energy between u and v due to dispersion and eventual elimination of the shock formation because of the presence of non-local term in the coupled system of non-local Burger's equations where non-locality introduces dispersion into the solution.

Results and Discussion

The following profiles are showing initial conditions for u and v under the law given by non-local coupled system and their evolution is given underneath.

$$\text{Kernel } K = -1 + \cos(2\pi x)$$

Initial Conditions:

$$u_0(x) = -0.84147 + \cos x$$

$$v_0(x) = +0.84147 - \cos x$$

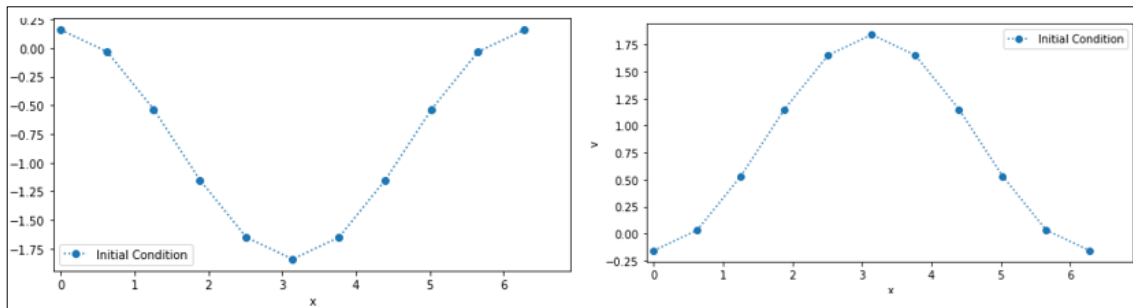


Fig 1: Initial Conditions for u and v

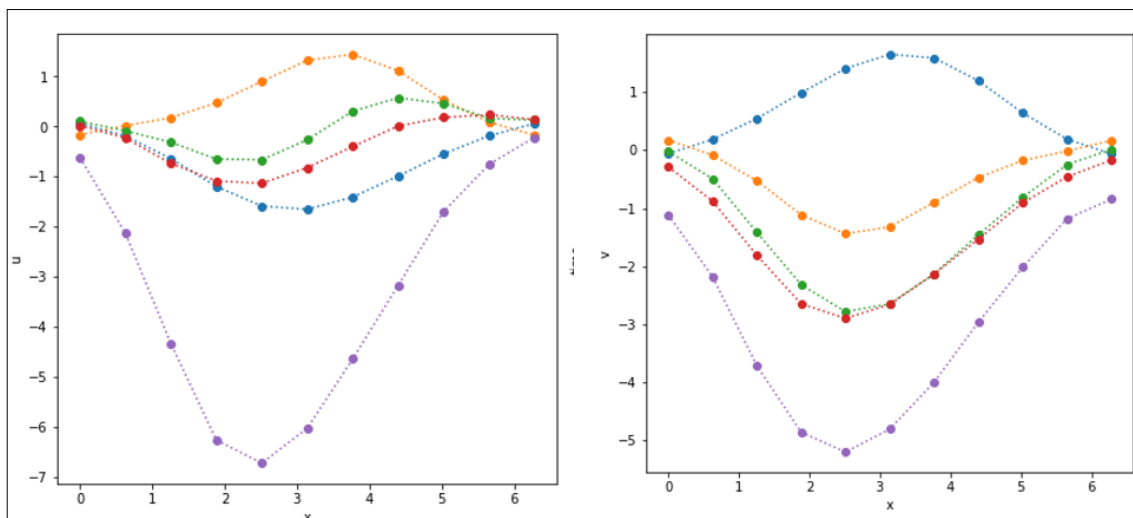


Fig 2: Evolved profiles of u and v

Figure 2 demonstrates how initial profiles evolve under the non-local Burger's system of equations. Profiles in Blue colour are the initial profiles. Profiles in orange colours arise the ones which are attained during the evolution where profiles are let to evolve for sufficiently longer time. Traditional system of Burger's equations admits shock wave solutions in general. We see in the above figure that non-linearity and dispersion compete to result in complete elimination of shock.

Conclusion

Resonant wave interactions form very important class of interactions where competition between non-linearity and dispersion leads to elimination of shock completely. However, this is not the only possibility in non-linear wave interactions. Cubic nonlinearity in the Burger's equation along with diffusion term leads to very peculiar class of solutions. In this type of equations, we find interaction between viscous term and capillarity term to produce non-classical shocks which are known as non-classical shocks since they don't obey classical entropy conditions. Kinematics approach is used in the analysis of these kind of equations. Viscosity and capillarity introduce two distinct scales in the solution and in some cases under-compressive shocks are accompanied by classical shocks. As Majda has pointed out in his article that detailed functional analytic analysis of the above kind of interactions is still in progress and authors propose to develop L^1 theory of above kind of interactions with techniques from compensated compactness methods so that non-linearity dispersion interaction and viscosity capillarity balance can be understood in its all generality. Much of the work in this direction is in progress and the topic is very promising to yield various other kinds of interactions using multiple scale techniques.

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