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Characteristic surfaces for coupled burger's equation

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Abstract

Solutions of non-linear hyperbolic partial differential equations are to a large extent governed by their characteristic surfaces viz, characteristic cone and normal cone. In the present article we have derived partial differential equation for characteristic surface for coupled system of Burger's equations in two space variables.

Keywords: Characteristic surface, normal cone, Burger's equation

Introduction

Before we present calculations of characteristic cone (Hilbert, 1962) ^[1] and normal cone for system comprised of generalized Burger's equations in two space variables it is apt to explain few notions from geometric theory of a scalar partial differential equation of first order. The equation which we are considering is the following.

$$F(t, x, y, u, p_1, p_2, p_3) = 0, p_i = \partial u / \partial x_i \quad (1)$$

In the above we have denoted t by x_1 , x by x_2 and y by x_3 for the convenience in notations. We associate with (1), the following system of ordinary differential equations.

$$\begin{aligned} \frac{dx_i}{ds} &= F_{p_i}, 1 \leq i \leq 3 \\ \frac{du}{ds} &= \sum_{i=1}^3 p_i F_{p_i}, 1 \leq i \leq 3 \\ \frac{dp_i}{ds} &= -(F_u p_i + F_{x_i}) 1 \leq i \leq 3 \end{aligned} \quad (2)$$

The above system of ordinary differential equations is for 7 unknowns, namely $t, x, y, u, p_1, p_2, p_3$, as a function of parameter s . The system (2) is known as the characteristic differential equations belonging to partial differential equation (1). The equation (1) is an integral of the system (2). All solutions of (2) are known as characteristic strips and these characteristic strips form a 5-parameter family. Infinitely many characteristic strips belong to the surface $u(t, x, y)$ which is a solution of (1). Every characteristic strip in common with the integral surface lies on the integral surface completely. System of differential equations which we are considering is of the general form stated below. Remember we will deal with the case $n = 3$ in what follows.

$$L[u] = \sum_{i=0}^n A^i \frac{\partial u}{\partial x_i} + Bu = f \quad (3)$$

$$lA = Ar = 0 \quad (4)$$

The surface represented by $Q = 0$ in ξ -space is known as characteristic surface for \eqref{systemgeneral}. Operator L is an interior differential operator on the characteristic surface in the sense that for data prescribed on the characteristic surface determine u only on

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the characteristic surface and no transversal component can be determined for u as data is prescribed only on the characteristic surface. Let $\phi(t, x) = \psi(x) - t = 0$ be a characteristic surface for (2) then $\phi(t) = -1, \phi_{x_i} = \psi_{x_i}$ for $i = 1, \dots, n$. Thus, in the discussion of (2) we have two spaces namely $x - t$ space and ξ space both are $n+1$ dimensional and there is a mechanism to identify $x - t$ space and ξ space with each other in certain circumstances. The normal velocity of the characteristic surface $\psi(x) = t$ is defined as a vector v orthogonal to ψ such that magnitude of v measures the rate of change of ψ along its normal. The variables v and ξ are dual to each other with respect to unit sphere. With this formulation characteristic condition can be written as follows-

$$Q(-|v|, \alpha_1, \dots, \alpha_n) = 0, \alpha = \frac{\xi}{|\xi|} \quad (5)$$

This equation is an algebraic equation for normal speed $|v|$ of the characteristic surface whose normal points in the direction α . With this formulation, for (2), the normal speeds v are the eigenvalues of the following matrix

$$\sum_{i=0}^n \alpha_i A^i \quad (6)$$

Normal to the characteristic surface element at a point O generate a cone and its equation is then

$$Q(\xi_0, \dots, \xi_n) = 0 \quad (7)$$

The above cone is known as the normal cone through O and for a fixed value of t , it is known as a normal surface and it belongs to the space comprised of $\xi_1, \dots, \xi_n$. Characteristics of first order partial differential equation $Q(D\phi) = 0$ are known as bicharacteristics. Bicharacteristics obey the following system of $2n+2$ ordinary differential equations.

$$\begin{aligned} \frac{dx_i}{ds} &= Q_{\xi_i} \\ \frac{d\xi_i}{ds} &= -Q_{x_i} \end{aligned} \quad (8)$$

Note that $Q(x, \xi)$ is an integral for the above system.

1. Burger's equation in Two Space Variables and its characteristic surface (Lax, 2005) ^[2]

$$\partial_t u + \frac{1}{2} \partial_x (u^2 + v^2) + \partial_y (uv) = 0$$

$$\partial_t v - \partial_x (uv) + \frac{1}{2} \partial_y (u^2 + v^2) = 0 \quad (9)$$

The above system is known as Burger's equation in two space variables. It is quasilinear system of conservation laws in two space dimensions which is equivalent to the following system of partial differential equations.

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \begin{pmatrix} v & u \\ -u & v \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (10)$$

The characteristic surface is given by:

$$\begin{vmatrix} \phi_t + u\phi_x + v\phi_y & v\phi_x + u\phi_y \\ -v\phi_x + u\phi_y & \phi_t - u\phi_x + v\phi_y \end{vmatrix} = 0 \quad (11)$$

On solving this determinant

$$\phi_t^2 - 2v\phi_t\phi_y - (u^2 - v^2)(\phi_x^2 + \phi_y^2) = 0 \quad (12)$$

The above first order partial differential equation give rise to characteristic surface for the Burger's equation in two space variables. Detailed analysis of this partial differential equation and its role in the propagation of singularities will be explained in the subsequent paper.

Conclusion

Understanding geometry of characteristic surfaces and its bicharacteristics is very important step in analysing propagation of singularities of systems of hyperbolic partial differential equations. This problem is extremely hard as characteristic surfaces have several irreducible components in general though inner sheet of the characteristic surface is always convex. Author has undertaken detailed investigation of characteristics surfaces and normal cones for general hyperbolic systems with multiple eigenmodes being equal. The work in this direction will gain pace if algebraic analysis of multisheeted surfaces in projective spaces will be within our reach.

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