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Characteristic polynomials of the matrices associated with equal-square graph

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Abstract

The Equal-Square graph of a finite group, $ES(G)$ is a finite, simple and undirected graph having pairs of vertices (u_1, u_2) adjacent iff $u_1^2 = u_2^2$. In this research, various matrices associated with $ES(G)$ and their characteristic polynomials are obtained.

Keywords: Equal-Square Graph, $ES(G)$, finite group, finite graph, simple graph

Introduction

In recent years, characteristic polynomials of various matrices of graphs of groups have been calculated. Various spectral and structural properties of square power graph are studied in [1, 2, 3, 4, 5]. Whereas, for cubic power graph of dihedral group in [6] and for finite abelian group in [7]. k^{th} power graph vertices degree is calculated in [8]. Various degree-based matrices characteristics polynomials are calculated in [9]. Various properties of equal-square graph are studied in [10].

In this research, we have calculated characteristics polynomials of maximum, minimum and greatest common divisor degree matrices of $ES(D_n)$.

Definition

[11] The maximum matrix of $ES(D_n)$, denoted by $Max(ES(D_n)) = [max_{ij}]_{2n \times 2n}$ whose $(i, j)^{th}$ entry is

$$max_{ij} = \begin{cases} \max\{deg_{ES(D_n)}(v_i), deg_{ES(D_n)}(v_j)\}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.2

[12] The minimum matrix of $ES(D_n)$, denoted by $Min(ES(D_n)) = [min_{ij}]_{2n \times 2n}$ whose $(i, j)^{th}$ entry is

$$min_{ij} = \begin{cases} \min\{deg_{ES(D_n)}(v_i), deg_{ES(D_n)}(v_j)\}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.3

[13] The greatest common divisor degree matrix of $ES(D_n)$, denoted by $GCDD(ES(D_n)) = [gcdd_{ij}]_{2n \times 2n}$ whose $(i, j)^{th}$ entry is

$$gcdd_{ij} = \begin{cases} \gcd\{deg_{ES(D_n)}(v_i), deg_{ES(D_n)}(v_j)\}, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1

^[14] Let $ES(D_n)$ be equal-square graph of D_n . Then, $ES(D_n) = K_{n+2} \cup (\frac{n-2}{2})K_2$, if n is even and $K_{n+1} \cup (n-1)K_1$ if n is odd.

Characteristic polynomials of $ES(G)$

Theorem 2.1 Let $P_{Max(ES(D_n))}$ be the characteristic polynomial of the maximum matrix of $ES(D_n)$. Then

1. $P_{Max(ES(D_n))} = (-1)^{\frac{3n}{2}} (n+1+\lambda)^{n+1} (1-\lambda^2)^{\frac{n-2}{2}} \{-\lambda + (n+1)^2\}$ when n is even.
2. $P_{Max(ES(D_n))} = \lambda^{n-1} (n+\lambda)^n (\lambda - n^2)$ when n is odd.

Proof. (i) When n is even.

Let O be the zero matrix of $(n+2) \times (n-2)$ order. Then, by using Theorem 1.1 maximum matrix of $ES(D_n)$ is given as

$$Max(ES(D_n)) = \begin{bmatrix} A_1 & O \\ O' & A_2 \end{bmatrix}_{2n \times 2n}$$

where,

$$A_1 = \begin{bmatrix} 0 & (n+1) & (n+1) & (n+1) & \cdots & (n+1) & (n+1) \\ (n+1) & 0 & (n+1) & (n+1) & \cdots & (n+1) & (n+1) \\ (n+1) & (n+1) & 0 & (n+1) & \cdots & (n+1) & (n+1) \\ (n+1) & (n+1) & (n+1) & 0 & \cdots & (n+1) & (n+1) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n+1) & (n+1) & (n+1) & (n+1) & \cdots & 0 & (n+1) \\ (n+1) & (n+1) & (n+1) & (n+1) & \cdots & (n+1) & 0 \end{bmatrix}_{(n+2)(n+2)}$$

$$\text{And } A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{(n-2)(n-2)}$$

Then,

$$P_{Max(ES(D_n))} = |Max(ES(D_n)) - \lambda I| = \begin{vmatrix} -\lambda & (n+1) & (n+1) & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ (n+1) & -\lambda & (n+1) & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ (n+1) & (n+1) & -\lambda & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n+1) & (n+1) & (n+1) & \cdots & -\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -\lambda \end{vmatrix}$$

Applying, $R_i \rightarrow R_i - R_1$ for $2 \leq i \leq n+2$ and $R_i \rightarrow R_i - R_{i-1}$ for $i = n+4, n+6, \dots, 2n$

We get,

$$|Max(ES(D_n)) - \lambda I| = \begin{vmatrix} -\lambda & (n+1) & (n+1) & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ (n+1) + \lambda & -\lambda - (n+1) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ (n+1) + \lambda & 0 & -\lambda - (n+1) & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n+1) + \lambda & 0 & 0 & \cdots & -\lambda - (n+1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 + \lambda & -\lambda - 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 + \lambda & -\lambda - 1 \end{vmatrix}$$

$$|Max(ES(D_n)) - \lambda I| = (n+1+\lambda)^{n+1} (1+\lambda)^{\frac{n-2}{2}} \times$$

$$\begin{vmatrix} -\lambda & (n+1) & (n+1) & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -1 \end{vmatrix}$$

Applying, $C_i \rightarrow C_i + C_{i+1}$ for $i = n+3, n+5, \dots, 2n-1$

$$|Max(ES(D_n)) - \lambda I| = (n+1+\lambda)^{n+1} (1+\lambda)^{\frac{n-2}{2}}$$

$$\begin{vmatrix} -\lambda & (n+1) & (n+1) & \cdots & (n+1) & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda+1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -\lambda+1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 \end{vmatrix}$$

After extending along $a_{(n+4)(n+4)}, a_{(n+6)(n+6)}, a_{(n+8)(n+8)}, \dots, a_{(2n)(2n)}$

$$|Max(ES(D_n)) - \lambda I| = (-1)^{\frac{n-2}{2}} (n+1+\lambda)^{n+1} (1+\lambda)^{\frac{n-2}{2}} (1-\lambda)^{\frac{n-2}{2}} \times$$

$$\begin{vmatrix} -\lambda & (n+1) & (n+1) & \cdots & (n+1) \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + \dots + C_{n+2}$

$$|Max(ES(D_n)) - \lambda I| = (-1)^{\frac{n-2}{2}} (n+1+\lambda)^{n+1} (1+\lambda)^{\frac{n-2}{2}} (1-\lambda)^{\frac{n-2}{2}} \times$$

$$\begin{vmatrix} -\lambda + (n+1)(n+1) & (n+1) & (n+1) & \cdots & (n+1) \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$P_{Max(ES(D_n))} = (-1)^{\frac{3n}{2}} (n+1+\lambda)^{n+1} (1-\lambda^2)^{\frac{n-2}{2}} \{-\lambda + (n+1)^2\}.$$

When n is odd.

Let O_1 and O_2 be the zero matrices of orders $(n+1) \times (n-1)$ and $(n-1) \times (n-1)$ respectively. Then, by using Theorem 1.1

maximum matrix of $ES(D_n)$ is given as $Max(ES(D_n)) = \begin{bmatrix} A_1 & O_1 \\ O_1' & O_2 \end{bmatrix}_{2n \times 2n}$

$$\text{where, } A_1 = \begin{bmatrix} 0 & n & n & n & \cdots & n & n \\ n & 0 & n & n & \cdots & n & n \\ n & n & 0 & n & \cdots & n & n \\ n & n & n & 0 & \cdots & n & n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & n & n & \cdots & 0 & n \\ n & n & n & n & \cdots & n & 0 \end{bmatrix}_{(n+1)(n+1)}$$

Then,

$$P_{Max(ES(D_n))} = |Max(ES(D_n)) - \lambda I| = \begin{vmatrix} -\lambda & n & n & \cdots & n & 0 & \cdots & 0 \\ n & -\lambda & n & \cdots & n & 0 & \cdots & 0 \\ n & n & -\lambda & \cdots & n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & -\lambda & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -\lambda \end{vmatrix}$$

Extending along $a_{(n+2)(n+2)}, a_{(n+3)(n+3)}, \dots, a_{(2n)(2n)}$

$$P_{Max(ES(D_n))} = \lambda^{n-1} \begin{vmatrix} -\lambda & n & n & \cdots & n \\ n & -\lambda & n & \cdots & n \\ n & n & -\lambda & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & -\lambda \end{vmatrix}$$

Applying, $R_i \rightarrow R_i - R_1$ for $2 \leq i \leq n+1$

$$\text{We get, } P_{Max(ES(D_n))} = \lambda^{n-1} \begin{vmatrix} -\lambda & n & n & \cdots & n \\ n+\lambda & -\lambda-n & 0 & \cdots & 0 \\ n+\lambda & 0 & -\lambda-n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+\lambda & 0 & 0 & \cdots & -\lambda-n \end{vmatrix}$$

$$= \lambda^{n-1} (n+\lambda)^n \begin{vmatrix} -\lambda & n & n & \cdots & n \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

Applying, $C_1 \rightarrow C_1 + C_2 + \cdots + C_{n+1}$

$$P_{Max(ES(D_n))} = \lambda^{n-1} (n+\lambda)^n \begin{vmatrix} -\lambda+n^2 & n & n & \cdots & n \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$P_{Max(ES(D_n))} = \lambda^{n-1} (n+\lambda)^n (\lambda - n^2).$$

Theorem 2.2 Let $(u_1, u_2) \in E(ES(D_n))$. Then, $degree(u_1) = degree(u_2)$.

Corollary 2.1 Let $P_{Min(ES(D_n))}$ and $P_{GCDD(ES(D_n))}$ be characteristic polynomials of minimum and greatest common divisor degree matrices of $ES(D_n)$ respectively. Then, $P_{Max(ES(D_n))} = P_{Min(ES(D_n))} = P_{GCDD(ES(D_n))}$.

Proof. Using Theorem 2.2, we have required result.

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