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A new kind of pseudo topological $\widetilde{bc}$ -manifold

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Abstract

This research aims to introduce and explore a new type of manifold space, paving the way for a more detailed study. We begin by defining a new open set ($\widetilde{bc}$ -open) based on filter convergence. Building on this foundation, we define the $\widetilde{bc}$ -chart and $\widetilde{bc}$ -atlas, assuming that the functions involved are $\widetilde{bc}$ -homeomorphisms. The study is then extended to cover functions that are $\widetilde{bc}$ -diffeomorphisms, following a precise definition of this pseudo $\widetilde{bc}$ -smooth manifold. Additionally, we examine functions between multiple manifolds and introduce the concepts of $\widetilde{bc}$ -covering spaces and $\widetilde{bc}$ -covering maps. Finally, we discuss various ideas related to the topic.

Keywords: $\widetilde{bc}$ -open, $\widetilde{bc}$ -chart, pseudo $\widetilde{bc}$ -manifold.

1. Introduction

Topological concepts, such as the concept of the filter, as it is used to describe pseudo topology and its properties, just as series are used to describe topology and its properties was first discussed by Henri Cartan in 1937 ^[1]. After that it was proposed by Bourbaki, N. in 1940 ^[2] and then by Choquet in ^[3] who used it to define convergence space and convergence structures, then after him Gähler in 1976 ^[4]. Pseudo topological spaces are one of the basic structures in mathematical analysis. The researcher in ^[5] defined the convergence structures and several concepts related to this space. After that, in ^[6, 7, 8, 9], the two researchers studied new properties of this space. In our research, we gave a new definition of the $\widetilde{bc}$ -open set and relied on it as a basic $\widetilde{bc}$ -structure to define the $\widetilde{bc}$ -chart, which in turn is considered the basic building block for building the pseudo $\widetilde{bc}$ -manifold. The concept introduced in our study can be effectively applied to a specific class of spaces, particularly fuzzy Fréchet spaces; see ^[10, 11, 12, 13]. These works have extensively examined the structural and functional properties of such spaces, providing a solid foundation for further theoretical developments. Building upon these insights, our study offers a new perspective that aligns with and extends the existing results in this area.

2. Preliminaries

Definition 2.1 ^[5] A filter in a real vector space M is a non-empty set $\mathcal{H}$ of subsets of M , such that: $\emptyset \notin \mathcal{H}$; $Z_1 \in \mathcal{H}$ and $Z_1 \subset Z_2 \rightarrow Z_2 \in \mathcal{H}$; $Z_1, Z_2 \in \mathcal{H} \rightarrow Z_1 \cap Z_2 \in \mathcal{H}$.

Definition 2.2 ^[5]: A limit structure (pseudo topology or p. t.) $\mathfrak{J}$ on M is map:

$(\mathfrak{J}: M \rightarrow 2^{\Omega(M)}), x \mapsto \mathfrak{J}(x)$. If $\mathcal{H} \in \mathfrak{J}(x)$, $\mathcal{H}$ converges to x in $\mathfrak{J}$ if every set $F \in \mathcal{H}$ contain x and written $\mathcal{H} \downarrow_x$ (Or $\mathcal{H} \in \mathfrak{J}(x)$) and for all filters $\mathcal{H}$ and $\mathbb{P}$:

- 1) $[x] \in \mathfrak{J}(x) \ni [x] = \{\mu \subset M, x \in \mu\}$.
- 2) $\mathcal{H} \in \mathfrak{J}(x), \mathcal{H} \subseteq \mathbb{P} \Rightarrow \mathbb{P} \in \mathfrak{J}(x)$.
- 3) $\mathcal{H} \in \mathfrak{J}(x), \mathbb{P} \in \mathfrak{J}(x) \Rightarrow \mathcal{H} \cap \mathbb{P} \in \mathfrak{J}(x)$.

Definition 2.3 ^[5]: A T_{pVS} is both a vector space (v.s.) and p.t. on it (The continuity of addition and multiplication maps).

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Definition 2.4 ^[5] Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} . The interior $\mathcal{A}$ of a set $\mathcal{A} \subseteq \mathcal{M}$ is explained by: $in(\mathcal{A}) = \{x \in \mathcal{A} : \mathcal{H} \in \mathfrak{S}(x) \text{ implies } \mathcal{A} \in \mathcal{H}\}$. We call it (open set) if $\mathcal{A} = in(\mathcal{A})$.

Definition 2.5 ^[5]: Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} and $\mathcal{A} \subseteq \mathcal{M}$, then the closure of $\mathcal{A}$ is $\bar{\mathcal{A}} = (CL \mathcal{A}) = \{x \in \mathcal{M} : \exists \mathbb{P} \in \mathfrak{S}(x), \mathcal{A} \in \mathbb{P}\}$. We can be called $\mathfrak{S}$ -closed set (for easily closed set) if $\mathcal{A} = CL \mathcal{A}$.

3. Main Results

Definition 3.1 Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} and $\mathcal{A} \subseteq \mathcal{E}$ then the set $\mathcal{A}$ is $\bar{b}$ -open if $\mathcal{A} \subseteq in(CL(\mathcal{A})) \cup CL(in(\mathcal{A}))$. And $BO(X)$ is the family of all $\bar{b}$ -open subset. And the set $\mathcal{A}$ is call $\mathbb{P}(\mathfrak{S}) \in \mathfrak{S}(\mathbb{P}(x))$ open if for each $y \in \mathcal{A} \in BO(X)$ there exists F (closed set) $\ni y \in F$.

Definition 3.2: Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} , it is called a separated, If $\mathcal{H} \in \mathfrak{S}(z)$ and $\mathcal{H} \in \mathfrak{S}(w)$, then $z = w$.

Definition 3.3 Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} . A $\bar{b}c$ -chart (or coordinate $\bar{b}c$ -chart) in Let $(\mathcal{M}, \mathfrak{S})$ be a T_{PVS} is a pair $(u, \mathfrak{f})$ where u is an $\bar{b}c$ -open subset of $\mathcal{M}$ and $\mathfrak{f}$ is a homeomorphism map of $u \mapsto \mathfrak{f}(u)$ $\bar{b}c$ -open subset of some $\mathcal{M}$.

For any $w \in \mathcal{M}$, a $\bar{b}c$ -chart $(u, \mathfrak{f})$, is a $\bar{b}c$ -chart at w if and only if $w \in u$, (if $\mathfrak{f}(w) = 0$, the $\bar{b}c$ -chart is centered at w).

Definition 3.4: A pseudo $\bar{b}c$ -manifold $\mathcal{M}$ (briefly $\mathcal{P}.\bar{b}c$ -m.) is a pseudo topological space modeled on R^n . (That's mean A pseudo topological space $\mathcal{M}$ is a pseudo $\bar{b}c$ -manifold if it a demits an $\bar{b}c$ -open covering $\{u_\alpha\}_{\alpha \in A}$ where each set u_α is homeomorphic to an $\bar{b}c$ -open subset of a module space R^n).

Definition 3.5: Let $(u_1, \mathfrak{f})$ and (u_2, β) are two $\bar{b}c$ -charts on $\mathcal{M} \ni u_1 \cap u_2 \neq \emptyset$ then the composite map $\beta \circ \mathfrak{f}^{-1}: \mathfrak{f}(u_1 \cap u_2) \rightarrow \beta(u_1 \cap u_2)$ (called $\bar{b}c$ -transition map from $\mathfrak{f}$ to β) is homeomorphism.

Definition 3.6: Two $\bar{b}c$ -charts $(u_1, \mathfrak{f})$, (u_2, β) of a $\mathcal{P}.\bar{b}c$ -m. are compatible if the two maps: $\beta \circ \mathfrak{f}^{-1}: \mathfrak{f}(u_1 \cap u_2) \rightarrow \beta(u_1 \cap u_2)$, $\mathfrak{f} \circ \beta^{-1}: \beta(u_1 \cap u_2) \rightarrow \mathfrak{f}(u_1 \cap u_2)$ are homeomorphism.

Definition 3.7: $\mathcal{A} = \{(z_m, \mathfrak{f}_m)\}_{m \in M}$ is defined as a collection of $\bar{b}c$ -charts on $\mathcal{M}$. $\mathcal{A}$ is call $\bar{b}c$ -atlas if satisfied: $\bigcup_{n \in M} z_n = \mathcal{M}$; The sets of the form $\mathfrak{f}_n(z_n \cap z_m)$ for $m, n \in M$ are all $\bar{b}c$ -open in R^n ; Whenever $z_n \cap z_m$ is not empty, the map $\mathfrak{f}_m \circ \mathfrak{f}_n^{-1}: \mathfrak{f}_n(z_n \cap z_m) \rightarrow \mathfrak{f}_m(z_n \cap z_m)$ is homeomorphism, are holds.

Definition 3.8: Let $\mathcal{M}$ be $\mathcal{P}.\bar{b}c$ -m.. A $\bar{b}c$ -chart $(v, \mathfrak{f})$ in $\mathcal{M}$ is known as $\tilde{C}^\infty$ - $\bar{b}c$ -chart if $\mathfrak{f}$ is a diffeomorphism map.

Example 3.9: Let $\mathcal{M}$ be $\mathcal{P}.\bar{b}c$ -m., and $A \subseteq \mathcal{M}$ $\bar{b}c$ -open set then, (A, e) is coordinate $\bar{b}c$ -chart in $\mathcal{M}$ such that e is identity map.

Definition 3.10: Let $(u_1, \mathfrak{f})$ and (u_2, β) are two $\bar{b}c$ -charts on $\mathcal{M} \ni u_1 \cap u_2 \neq \emptyset$, then the composite map $\beta \circ \mathfrak{f}^{-1}: \mathfrak{f}(u_1 \cap u_2) \rightarrow \beta(u_1 \cap u_2)$ is called $\bar{b}c$ -transition map from $\mathfrak{f}$ to β .

Definition 3.11: Let $(u_1, \mathfrak{f})$ and (u_2, β) are smooth compatible if the maps; $\beta \circ \mathfrak{f}^{-1}: \mathfrak{f}(u_1 \cap u_2) \rightarrow \beta(u_1 \cap u_2)$ and $\mathfrak{f} \circ \beta^{-1}: \beta(u_1 \cap u_2) \rightarrow \mathfrak{f}(u_1 \cap u_2)$ are smooth maps.

Definition 3.12: $\mathcal{A} = \{(z_j, \mathfrak{f}_j)\}_{j \in J}$ is defined as a collection of $\tilde{C}^\infty$ - $\bar{b}c$ -charts on $\mathcal{M}$. $\mathcal{A}$ is known as $\tilde{C}^\infty$ - $\bar{b}c$ -atlas :

- 1- $\bigcup_{i \in I} z_i = \mathcal{M}$.
- 2- $\mathfrak{f}_i(z_i \cap z_j)$ are $\bar{b}c$ -open $\in R^n, \forall i, j \in I$.
- 3- Every time $z_i \cap z_j \neq \emptyset$, then $\mathfrak{f}_j \circ \mathfrak{f}_i^{-1}: \mathfrak{f}_i(z_i \cap z_j) \rightarrow \mathfrak{f}_j(z_i \cap z_j)$ is a diffeomorphism. Are confirmed.

Definition 3.13: $\mathcal{A}$ and $\mathcal{A}_1$ are compatible $\tilde{C}^\infty$ - $\bar{b}c$ -atlases on $\mathcal{M}$ when $\tilde{C}^\infty$ - $\bar{b}c$ -chart of any is compatible with the other atlas (or $\mathcal{A} \cup \mathcal{A}_1$ is also smooth $\bar{b}c$ -atlas). And a $\tilde{C}^\infty$ - $\bar{b}c$ -chart $(z, \mathfrak{f})$ is compatible with $\tilde{C}^\infty$ - $\bar{b}c$ -atlas $\mathcal{A} = \{(z_j, \mathfrak{f}_j)\}_{j \in J}$ if and only if $\mathcal{A} \cup \{(z, \mathfrak{f})\}$ is a $\tilde{C}^\infty$ - $\bar{b}c$ -atlas.

Definition 3.14: If every $\tilde{C}^\infty$ - $\bar{b}c$ -chart that is $\tilde{C}^\infty$ -compatible with all $\tilde{C}^\infty$ - $\bar{b}c$ -chart in $\mathcal{A}$ then a $\tilde{C}^\infty$ - $\bar{b}c$ -atlas $\mathcal{A}$ on $\mathcal{M}$ is maximal.

Remark 3.15: Two smooth structures $\tilde{A}_1$ and $\tilde{A}_2$ on the same $\mathcal{P}.\bar{b}c$ -m. $\mathcal{M}$ are said to be different if (Identity map) $Id: (\mathcal{M}; \tilde{A}_1) \rightarrow (\mathcal{M}; \tilde{A}_2)$ is not a diffeomorphism.

Definition 3.16: A $\mathcal{P}.\bar{b}c$ -m. of class $\tilde{C}^\infty$ is $(\mathcal{M}, \mathcal{A})$ where $\mathcal{M}$ is a $\mathcal{P}.\bar{b}c$ -m. and $\mathcal{A}$ is a smooth structure ($\tilde{C}^\infty$ -structure) on $\mathcal{M}$.

Example 3.17: Let $E \neq \emptyset$ be open set in some Euclidean space then E is the union of countably many compact sets $\mathcal{K}_n \neq \emptyset$ such that $\mathcal{K}_n \subset \mathcal{K}_{n+1}, \forall n \in \mathbb{N} - \{0\}$, $\mathcal{C}(E)$ is the vector space of all complex valued continuous functions on E , topologized by the separating family of PSN: $\|f\|_n = \sup\{|\mathfrak{f}(x)| : x \in \mathcal{K}_n\}$, since $\mathbb{P} = \{\|\cdot\|_i\}_{i \in I}$ is non-decreasing. The sets $\mathcal{V}_n(\|\cdot\|_i) = \{f \in \mathcal{C}(E) : \|f\|_n < \frac{1}{n}\}, n \in \mathbb{N} - \{0\}$ form a convex local base for $\mathcal{C}(E)$, the pseudo topology of $\mathcal{C}(E)$ is compatible with the pseudo metric $d(f, g) = \max_n \frac{2^{-n} \|f\|_n}{1 + \|f\|_n}$, $\mathcal{C}(E)$ with compatible pseudo topology τ is a T_{PVS} . Let $\mathcal{M} = \mathcal{C}(E)$ be a pseudo topological vector space. We defined $\tilde{C}^\infty$ - $\bar{b}c$ -chart (A, ϑ) on $\mathcal{M}$ as follows: $A = \mathcal{M}$ and ϑ is identity map, $\vartheta(m) = m, \forall m \in \mathcal{M}$. Then, (A, ϑ) satisfies the conditions of $\tilde{C}^\infty$ - $\bar{b}c$ -chart. Let $\mathcal{A} = \{(A_i, \vartheta_i)\}_{i \in I}$ be atlas on pseudo topological vector space $\mathcal{M}$, $\mathcal{A}$ is maximal $\tilde{C}^\infty$ -atlas on $\mathcal{M}$.

Therefore, $(\mathcal{M}, \mathcal{A})$ is pseudo $\bar{b}c$ -manifold of class $\tilde{C}^\infty$.

Proposition 3.18: Let $\mathcal{M}$ be a $\mathcal{P}.\bar{b}c$ -m. of class $\tilde{C}^\infty$, and let (u_1, α_1) be a smooth compatible with (u_2, α_2) , and also (u_2, α_2) is smooth compatible with (u_3, α_3) in a $\tilde{C}^\infty$ - $\bar{b}c$ -atlas $\mathcal{A}$, then (u_1, α_1) be a smooth compatible with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

Proof: Since (u_1, α_1) be a smooth compatible with (u_2, α_2) then the map

$\alpha_2 \circ \alpha_1^{-1}: \alpha_1(u_1 \cap u_2) \rightarrow \alpha_2(u_1 \cap u_2)$ is smooth and also, since (u_2, α_2) be a smooth compatible with (u_3, α_3) then the map $\alpha_3 \circ \alpha_2^{-1}: \alpha_2(u_2 \cap u_3) \rightarrow \alpha_3(u_2 \cap u_3)$ is smooth, and the $\bar{b}c$ -open set $u_1 \cap u_2 \cap u_3$ is subset of $u_1 \cap u_2$ and subset of $u_2 \cap u_3$, and the map $\alpha_3 \circ \alpha_1^{-1} = (\alpha_3 \circ \alpha_2^{-1}) \circ (\alpha_2 \circ \alpha_1^{-1})$ is composite maps of class $\tilde{C}^\infty$ then its map of class $\tilde{C}^\infty$ on $\alpha_1(u_1 \cap u_2 \cap u_3)$, hence (u_1, α_1) be compatible of class $\tilde{C}^\infty$ with (u_3, α_3) on $\alpha_1(u_1 \cap u_2 \cap u_3)$.

Lemma 3.19: Let $\mathcal{A} = \{(z_j, \mathcal{f}_j)\}_{j \in J}$ be $\tilde{C}^\infty$ - $\widetilde{bc}$ -atlas on a $\mathcal{P}.$ $\widetilde{bc}$ -m. $\mathcal{M}$. If two $\tilde{C}^\infty$ - $\widetilde{bc}$ -charts (u, ϑ) , (ω, γ) are compatible of $\tilde{C}^\infty$ with the $\tilde{C}^\infty$ - $\widetilde{bc}$ -atlas, $\mathcal{A} = \{(z_j, \mathcal{f}_j)\}_{j \in J}$, then they are compatible of $\tilde{C}^\infty$ with each other.

Proof: Let $z \in u \cap \omega$. We will prove that $\gamma \circ \vartheta^{-1}$ is $\tilde{C}^\infty$ at $\vartheta(z)$. Since $\mathcal{A} = \{(u_\delta, \xi_\delta)\}$ is an $\tilde{C}^\infty$ - $\widetilde{bc}$ -atlas in $\mathcal{M}$, $z \in u_\delta$ for some δ . Then, $z \in u \cap \omega \cap u_\delta$. From above the proposition, $\gamma \circ \vartheta^{-1} = (\gamma \circ \mathcal{f}_\delta^{-1}) \circ (\mathcal{f}_\delta \circ \vartheta^{-1})$ is smooth on $\vartheta(u \cap \omega \cap u_\delta)$, hence at $\vartheta(z)$. Since z is an arbitrary point of $u \cap \omega$, then $\gamma \circ \vartheta^{-1}$ is smooth on $\vartheta(u \cap \omega)$. Similarly, $\vartheta \circ \gamma^{-1}$ is smooth on $\gamma(u \cap \omega)$. Therefore, (u, ϑ) and (ω, γ) are compatible of $\tilde{C}^\infty$.

Proposition 3.20: Let $\omega \subset \mathcal{M}$ be $\widetilde{bc}$ -open set. If $\mathcal{f}: \omega \rightarrow \mathcal{f}(\omega) \subset \mathbb{R}^n$ is a diffeomorphism. Then $(\omega, \mathcal{f})$ is $\tilde{C}^\infty$ $\widetilde{bc}$ -chart in $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas of $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$ $\mathcal{M}$.

Proof: Let $(v_a, \mathcal{f}_a)$ be a $\tilde{C}^\infty$ $\widetilde{bc}$ -chart in $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas of $\mathcal{M}$, then $\mathcal{f} \circ \mathcal{f}_a^{-1}$ and $\mathcal{f}_a \circ \mathcal{f}^{-1}$ will be $\tilde{C}^\infty$. And $(v, \mathcal{f})$ is $\tilde{C}^\infty$ compatible with the $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas. By maximality of $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas of $\mathcal{M}$, $\tilde{C}^\infty$ $\widetilde{bc}$ -chart $(v, \mathcal{f})$ is in $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas of $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$ $\mathcal{M}$.

Definition 3.21: Let $f: \mathcal{M} \rightarrow \mathbb{R}$ $\tilde{C}^\infty$ function ($\tilde{C}^\infty$ differentiable) on $\mathcal{M}$ if the function $f \circ \mathcal{f}^{-1}: w \rightarrow \mathbb{R}$ is $\tilde{C}^\infty$ such that $\mathcal{f}: u \rightarrow w \subset \mathbb{R}^n$.

Definition 3.22: A map $\mathbb{Y}: U \rightarrow V$, $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ is said to be $\tilde{C}^\infty$ if each of the component map of $\mathbb{Y}$ has continuous partial derivatives of all orders.

Definition 3.23: Let $\mathcal{M}$ and N be $\mathcal{P}.$ $\widetilde{bc}$ -m. of class $\tilde{C}^\infty$ on a $\mathbb{R}^n$, and let $\mathbb{Y}: \mathcal{M} \rightarrow N$ be a map. For any $s \in \mathcal{M}$ has μ (neighborhood) $\ni$ the restriction $\mathbb{Y}|_\mu$ is smooth iff $\mathbb{Y}$ is smooth.

Lemma 3.24: Let, $\{(u_n, \theta_n)\}_{n \in \mathbb{N}}$ is a $\tilde{C}^\infty$ - $\widetilde{bc}$ -atlas for $\mathcal{B} = \mathbb{C}$. If $f: \mathcal{M} \rightarrow \mathbb{R}^m$ is a map $\ni f \circ \theta_n^{-1}$ is smooth $\forall n$, f is smooth.

Proof: $f \circ \theta_n^{-1}$ is to be proved as smooth for every $\tilde{C}^\infty$ - $\widetilde{bc}$ -chart (u, θ) on $\mathcal{M}$. It is enough for us to prove that it is smooth in a neighborhood of any point $y = \theta(x_0) \in \theta(u)$. For any $y_0 \in u$, $\exists$ $\widetilde{bc}$ -chart $(u_n, \theta_n) \in \tilde{C}^\infty$ - $\widetilde{bc}$ -atlas ($y_0 \in \text{domain}$ it). If (u, θ) is smoothly compatible with (u_n, θ_n) . $\theta_n \circ \theta^{-1}$ is $\tilde{C}^\infty$ on its domain of definition ($y \in \text{domain}$ it). So $f \circ \theta^{-1} = (f \circ \theta_n^{-1}) \circ (\theta_n \circ \theta^{-1})$ is $\tilde{C}^\infty$ in a neighborhood of y .

Lemma 3.25: If $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ and $\sigma: \mathcal{M}_2 \rightarrow \mathcal{M}_3$ are $\tilde{C}^\infty$ maps. $\sigma \circ \mathcal{E}$ is $\tilde{C}^\infty$ map.

Proof: Let (u, θ) and (v, ϑ) are $\tilde{C}^\infty$ - $\widetilde{bc}$ -charts for $\mathcal{M}_1$ and $\mathcal{M}_3$ respectively. To prove $\sigma \circ \mathcal{E}$ is smooth map $(\vartheta \circ (\sigma \circ \mathcal{E}) \circ \theta^{-1})$ is smooth on its domain of definition, specifically on $\theta(u \cap (\sigma \circ \mathcal{E})^{-1}(v))$. Let $s \in u \cap (\sigma \circ \mathcal{E})^{-1}(v)$, $\exists$ a $\widetilde{bc}$ -chart (z, μ) for $\mathcal{M}_2 \ni \mathcal{E}(x) \in z$. By smoothness of $\mathcal{E}$ and $\sigma \Rightarrow \mu \circ \mathcal{E} \circ \theta^{-1}$ and $\vartheta \circ \sigma \circ \mu^{-1}$ are $\tilde{C}^\infty$ on its domain of definition $\Rightarrow \vartheta \circ (\sigma \circ \mathcal{E}) \circ \theta^{-1} = (\vartheta \circ \sigma \circ \mu^{-1}) \circ (\mu \circ \mathcal{E} \circ \theta^{-1})$ is smooth.

Lemma 3.26: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be any $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$, the

space $\mathcal{M}_1 \times \mathcal{M}_2$ has smooth $\widetilde{bc}$ -structure $\ni$ the projection maps $\eta_1: \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{M}_1$, $\eta_2: \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow \mathcal{M}_2$ are smooth. $\mathcal{M}_1 \times \mathcal{M}_2$ is called the product $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$.

Proof: Suppose $\{(Z_i, \mathcal{h}_i)_{i \in I}\}$, $\{(J_j, \mathcal{g}_j)_{j \in J}\}$ that $\tilde{C}^\infty$ $\widetilde{bc}$ -atlases for a two $\mathcal{P}.$ $\widetilde{bc}$ -m. $\mathcal{M}_1, \mathcal{M}_2$ respectively. So $\{(Z_i \times J_j)_{(i,j) \in I \times J}\}$ is $\widetilde{bc}$ -open cover of $\mathcal{M}_1 \times \mathcal{M}_2$ and the $\widetilde{bc}$ -coordinate $\tilde{C}^\infty$ map $\mathcal{h}_i \times \mathcal{g}_j: Z_i \times J_j \rightarrow \mathcal{h}_i(Z_i) \times \mathcal{g}_j(J_j)$ define a $\tilde{C}^\infty$ $\widetilde{bc}$ -atlas. As far as $\widetilde{bc}$ -structure is concerned, its compatibility and smoothness are confirmed, then $\mathcal{M}_1 \times \mathcal{M}_2$ is $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$.

Definition 3.27: $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a local diffeomorphism if $\forall x \in \mathcal{M}_1$ has a neighborhood $u \ni \mathcal{E}(u)$ is $\widetilde{bc}$ -open in $\mathcal{M}_2$ and $\mathcal{E}: u \rightarrow \mathcal{E}(u)$ is a diffeomorphism.

Definition 3.28: Let $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$, a smooth $\widetilde{bc}$ -covering map of $\tilde{C}^\infty$, $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a $\tilde{C}^\infty$ bijective map $\ni \forall s \in \mathcal{M}_2$ has a neighborhood $\mu \ni$ each component of $\mathcal{E}^{-1}(\mu)$ is diffeomorphic ally on to μ by $\mathcal{E}$, we say that μ is evenly $\widetilde{bc}$ -covered. A $\mathcal{P}.$ $\widetilde{bc}$ -m. $\mathcal{M}_2$ is called a base to $\widetilde{bc}$ -covering, and $\mathcal{M}_1$ is called a $\widetilde{bc}$ -covering space of $\mathcal{M}_1$.

Definition 3.29: Let $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be continuous, a $(bc)^\sim$ -section of $\mathcal{E}$ is a continuous map $\mathcal{E}: \mathcal{M}_2 \rightarrow \mathcal{M}_1 \ni \mathcal{E} \circ \mathcal{E} = id_{\mathcal{M}_2}$, ($id_{\mathcal{M}_2}$ is identity map on $\mathcal{M}_2$).

Definition 3.30: Let $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be continuous map, a local $(bc)^\sim$ -section is a continuous map $\mathcal{E}: \mu \rightarrow \mathcal{M}_1$ defined on $\widetilde{bc}$ -open set $\mu \subset \mathcal{M}_2$ and satisfy a condition $\mathcal{E} \circ \mathcal{E} = id_\mu$.

Lemma 3.31: Let $\mathcal{E}: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a $\tilde{C}^\infty$ $\widetilde{bc}$ -covering map. For all point of $\mathcal{M}_1$ is in the image of a $\tilde{C}^\infty$ local $(bc)^\sim$ -section of $\mathcal{E}$. For any $r \in \mathcal{M}_1$, $\exists$ a neighbourhood μ of $s = \mathcal{E}(r)$ and a $\tilde{C}^\infty$ local $(bc)^\sim$ -section $\rho: \mu \rightarrow \mathcal{M}_1$ such that $\rho(s) = r$.

Proof: Let $\mu \subset \mathcal{M}_2$ be an evenly $\widetilde{bc}$ -covered neighbourhood of s . If v is the component of $\mathcal{E}^{-1}(\mu)$ and $r \in v$, then by hypothesis, $\mathcal{E}|_v: v \rightarrow \mu$ is a diffeomorphism. Then, $\rho = (\mathcal{E}|_v)^{-1}: \mu \rightarrow v$ is a $\tilde{C}^\infty$ local $\widetilde{bc}$ -section of $\mathcal{E}$ and such that, $\mathcal{E}(s) = r$.

Proposition 3.32: Let $\sigma: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a $\widetilde{bc}$ -covering $\tilde{C}^\infty$ map and $\mathcal{M}_3$ is any $\mathcal{P}.$ $\widetilde{bc}$ -m. of $\tilde{C}^\infty$. $\mathcal{F}: \mathcal{M}_2 \rightarrow \mathcal{M}_3$ is $\tilde{C}^\infty$ map iff $\mathcal{F} \circ \sigma: \mathcal{M}_1 \rightarrow \mathcal{M}_3$ is $\tilde{C}^\infty$:

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{\sigma} & \mathcal{M}_2 \\ \mathcal{F} \circ \sigma \searrow & & \downarrow \mathcal{F} \\ & & \mathcal{M}_3 \end{array}$$

Composition of Smooth $\widetilde{bc}$ -Covering Maps.

Proof: The evidence for the first trend is clear. Let $\mathcal{F} \circ \rho$ be smooth, and $r \in \mathcal{M}_2$ (is arbitrary). $\exists$ neighborhood u of r and $\tilde{C}^\infty$ local $\widetilde{bc}$ -section $\sigma: u \rightarrow \mathcal{M}_1$, so that $\rho \circ \sigma = id_u$. The restriction of $\mathcal{F}$ to u achieves: $\mathcal{F}|_u = \mathcal{F} \circ id_u = \mathcal{F} \circ \rho \circ \sigma = id_u$.

$(\rho \circ \sigma) = (F \circ \rho) \circ \sigma$ is a composition of $\tilde{C}^\infty$. So F is $\tilde{C}^\infty$ on u . And so F is $\tilde{C}^\infty$ in a neighborhood for all points.

Proposition 3.33: If $\mathcal{M}_2$ is a $\mathcal{P}.\widetilde{bc}$ -m. of $\tilde{C}^\infty$ and $\sigma: \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is any p.t. $\widetilde{bc}$ -covering map, then $\mathcal{M}_1$ has a $\tilde{C}^\infty \mathcal{P}.\widetilde{bc}$ -m. structure $\exists \sigma$ is a $\tilde{C}^\infty \widetilde{bc}$ -covering map.

Proof: Let s, r be a two deferent points in $\mathcal{M}_1$. Suppose $\mathfrak{I}(s) \cap \mathfrak{I}(r) \neq \emptyset$ then $\exists$ a filter $\mathfrak{F}$ in $\mathfrak{I}(s) \cap \mathfrak{I}(r)$, hence $\mathfrak{F} \in \mathfrak{I}(s)$ and $\mathfrak{F} \in \mathfrak{I}(r)$ on $\mathcal{M}_1$. Then, $\sigma(\mathfrak{F}) \in \mathfrak{I}(\sigma(s))$ and $\sigma(\mathfrak{F}) \in \mathfrak{I}(\sigma(r))$, since $\mathcal{M}_2$ is separated then, $\sigma(s) = \sigma(r)$. From infectivity of σ we get the contradiction. So, $\mathfrak{I}(s) \cap \mathfrak{I}(r) = \emptyset$. So $\mathcal{M}_1$ is separated. Let $s \in \mathcal{M}_2$ be evenly $\widetilde{bc}$ -covered neighbourhood. We shrink u (if necessary), and we impose it is the domain of a $\widetilde{bc}$ -coordinate map $\mathfrak{Q}: u \rightarrow R$. If ν is a component of $\sigma^{-1}(u)$ and $\tilde{\mathfrak{Q}} = \mathfrak{Q} \circ \sigma: \tilde{u} \rightarrow R$, then $(\tilde{u}, \tilde{\mathfrak{Q}})$ is a $\widetilde{bc}$ -chart on $\mathcal{M}_2$. If the two $\widetilde{bc}$ -charts $(\tilde{u}, \tilde{\mathfrak{Q}})$ and $(\tilde{\nu}, \tilde{\varphi})$ overlap, the transition map it written as $\tilde{\varphi} \circ \tilde{\mathfrak{Q}}^{-1} = (\varphi \circ \sigma /_{\tilde{u} \cap \tilde{\nu}}) \circ (\mathfrak{Q} \circ \sigma /_{\tilde{u} \cap \tilde{\nu}})^{-1} = \varphi \circ \sigma /_{\tilde{u} \cap \tilde{\nu}} \circ (\sigma /_{\tilde{u} \cap \tilde{\nu}})^{-1} \circ \mathfrak{Q}^{-1} = \varphi \circ \mathfrak{Q}^{-1}$ which is $\tilde{C}^\infty$.

4. Conclusion

The research presented a new type of pseudo- $\widetilde{bc}$ -manifolds after defining the $\widetilde{bc}$ -open set in the pseudo topological space in a new way. We also dealt with the study of functions defined on this pseudo-manifold and the Euclidean space, homomorphism functions, and then we expanded it to be diffeomorphism functions. We also studied the functions between pseudo- $\widetilde{bc}$ -manifolds, and finally we studied the concepts of $\widetilde{bc}$ -coverage space and $\widetilde{bc}$ -coverage functions.

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