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## Weighted composition operators on Bicomplex bloch space

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### Abstract

Let  $\mathbb{U}$  be the bidisk in  $\mathbb{BC}$  and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  as well as  $f: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . For a bicomplex holomorphic map  $f$  on  $\mathbb{U}$  the weighted composition operator  $u \cdot C_\phi$  is defined by  $(u \cdot C_\phi f)(z) = u(z) \cdot f(\phi(z))$  for every  $z \in \mathbb{U}$ , where  $u$  be a fixed bicomplex holomorphic function on  $\mathbb{U}$ . The present study deals the condition for boundedness and compactness of Weighted Composition Operators on bicomplex Bloch space which define over the unit ball  $\mathbb{U}$ , an extension of the classical Bloch space. The bicomplex Bloch is a higher-dimensional generalization of the Bloch space, structured to accommodate bicomplex analysis. The bicomplex Bloch space over the unit ball  $\mathbb{U}$  is useful for studying more advanced problems in analysis.

**Keywords:** Bicomplex, studying, problems

### Introduction

The theory of bicomplex numbers is an area of active research for quite a long time since the innovative work of Segre in search of special algebras. In 1892, Segre proposed the concept of bicomplex numbers which can be thought of as a generalization of complex numbers. Because not every non zero bicomplex number has a multiplicative inverse, the set of bicomplex numbers is a commutative ring with unity that contains the field of complex numbers but does not form a field, the study of zero divisors in bicomplex analysis is introduced. For a long time, bicomplex numbers have been explored. In recent years, a large amount of work has been published. G. B. Price's work <sup>[8]</sup> is a thorough examination of bicomplex numbers. One can refer to <sup>[1, 5, 8]</sup> for a systematic study of basic of bicomplex functional analysis.

Composition operators are extensively studied on various holomorphic function spaces of different domains in  $\mathbb{C}$  and  $\mathbb{C}^n$ . There are many papers on the study of composition operators on Bloch space and Bloch type spaces of holomorphic functions, see <sup>[2, 6]</sup> and references therein. The boundedness and the compactness of weighted composition operators on Bloch space was characterised by Ohno, see <sup>[7]</sup>. My results can be viewed as a generalisations of their results.

### Preliminaries

In this section, we summarize some common facts about bicomplex numbers and bicomplex holomorphic functions. The set of bicomplex numbers  $\mathbb{BC}$  is defined as.

$$\mathbb{BC} = \{Z = w_1 + jw_2 | w_1, w_2 \in \mathbb{C}(i)\},$$

where  $i$  and  $j$  are imaginary units such that  $ij = ji, i^2 = j^2 = -1$  and  $\mathbb{C}(i)$  is the set of complex numbers with the imaginary unit  $i$ . The set  $\mathbb{BC}$  of bicomplex numbers form a ring under the usual addition and multiplication of bicomplex numbers. " Moreover,  $\mathbb{BC}$  is a module over itself. The set of positive hyperbolic number is denoted by  $\mathbb{D}^+$  which is a subset of set of hyperbolic numbers  $\mathbb{D}$  is given by

$$\mathbb{D}^+ = \{\beta_1 + k\beta_2: \beta_1^2 - \beta_2^2 \geq 0, \beta_1 \geq 0\}.$$

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We can discuss three conjugations for bicomplex numbers in the same way we can do for usual complex numbers, because  $\mathbb{BC}$  comprises two imaginary units with squares equal to -1 and a hyperbolic units with square equal to 1.

1.  $\bar{Z} = \bar{w}_1 + j\bar{w}_2$  (the bar – conjugation);
2.  $Z^\dagger = w_1 - jw_2$ , (the  $\dagger$  – conjugation);
3.  $Z^* = \bar{w}_1 - j\bar{w}_2$ , (the  $*$  – conjugation),

where  $\bar{w}_1, \bar{w}_2$  denote the usual complex conjugates to  $w_1, w_2 \in \mathbb{C}(i)$ .

If  $Z = w_1 + jw_2 \neq 0$  is such that  $Z \cdot Z^\dagger = w_1^2 + w_2^2 = 0$ , then  $Z$  is a zero divisor. The set of zero divisors  $\mathbb{NC}$  of  $\mathbb{BC}$  is, thus, given by

$$\mathbb{NC} = \{Z \mid Z \neq 0, w_1^2 + w_2^2 = 0\},$$

and is called the null cone and  $\mathbb{NC}_0 = \mathbb{NC} \cup \{0\}$ .

The hyperbolic numbers  $e$  and  $e^\dagger$  defined as

$$e = \frac{1+k}{2} \text{ and } e^\dagger = \frac{1-k}{2},$$

are zero divisors, which are linearly independent in the  $\mathbb{C}(i)$ - vector space and satisfy the following properties:

$$e^2 = e, (e^\dagger)^2 = e^\dagger, e^* = e, (e^\dagger)^* = e^\dagger, e + e^\dagger = 1, e \cdot e^\dagger = 0.$$

Any bicomplex number  $Z = w_1 + jw_2$  can be uniquely written as "

$$Z = z_1 e + z_2 e^\dagger \tag{2.1}$$

where  $z_1 = w_1 - iw_2$  and  $z_2 = w_1 + iw_2$  are elements of  $\mathbb{C}(i)$ . Formula (2.1) is called the idempotent representation of a bicomplex number  $Z$ . A hyperbolic number  $\alpha = \beta_1 + k\beta_2$  in idempotent representation can be written as

$$\alpha = \alpha_1 e + \alpha_2 e^\dagger,$$

where  $\alpha_1 = \beta_1 + \beta_2$  and  $\alpha_2 = \beta_1 - \beta_2$  are real numbers. We say that  $\alpha$  is a positive hyperbolic number if  $\alpha_1 \geq 0$  and  $\alpha_2 \geq 0$ . Writing these hyperbolic numbers in their idempotent form  $\alpha = \alpha_1 e + \alpha_2 e^\dagger$  and  $\gamma = \gamma_1 e + \gamma_2 e^\dagger$ , with real numbers  $\alpha_1, \alpha_2, \gamma_1$  and  $\gamma_2$ , we have that

$$\alpha \leq \gamma \text{ iff } \alpha_1 \leq \gamma_1 \text{ and } \alpha_2 \leq \gamma_2.$$

If  $\gamma - \alpha \in \mathbb{D}^+ \setminus \{0\}$ , we write  $\gamma > \alpha$ . This implies that  $z \in \mathbb{D}^+$  is equivalent to  $z \geq 0$  and that  $z \in \mathbb{D}^+ \setminus \{0\}$  is equivalent to  $z > 0$ . A module defined over the ring of bicomplex numbers  $\mathbb{BC}$  (or ring of hyperbolic numbers  $\mathbb{D}$ ) is called a  $\mathbb{BC}$ -module (or  $\mathbb{D}$ -module). Consider the set  $\mathfrak{X}_1 = \mathfrak{X}e$  and  $\mathfrak{X}_2 = \mathfrak{X}e^\dagger$ . Then  $\mathfrak{X}_1 \cap \mathfrak{X}_2 = \{0\}$ . Thus, we can write

$$\mathfrak{X} = \mathfrak{X}_1 e + \mathfrak{X}_2 e^\dagger, \tag{2.2}$$

where  $\mathfrak{X}_1 = \mathfrak{X}e$  and  $\mathfrak{X}_2 = \mathfrak{X}e^\dagger$  are  $\mathbb{C}(i)$ -vector (or  $\mathbb{R}$ -vector) spaces. Equation (2.2) is called the idempotent decomposition of  $\mathfrak{X}$ .

**Definition 2.1** Let  $\mathfrak{X}$  be a  $\mathbb{BC}$  module. A function  $\|\cdot\|_{\mathbb{D}}: \mathfrak{X} \rightarrow \mathbb{D}^+$  is said to be a hyperbolic-valued norm (or  $\mathbb{D}$ -valued norm) on  $\mathfrak{X}$  if it satisfies the following properties:

1.  $\|x\|_{\mathbb{D}} = 0$  if and only if  $x = 0$ .
2.  $|\mu x|_{\mathbb{D}} = |\mu|_k \|x\|_{\mathbb{D}}, \forall x \in \mathfrak{X}, \forall \mu \in \mathbb{BC}$ .
3.  $\|x + y\|_{\mathbb{D}} \leq \|x\|_{\mathbb{D}} + \|y\|_{\mathbb{D}}, \forall x, y \in \mathfrak{X}$ .

The  $\mathbb{BC}$  module can be endowed canonically with the hyperbolic, or  $\mathbb{D}$ -valued norm denoted by  $\|\cdot\|_{\mathbb{D}}$  as follows:

$$\|x\|_{\mathbb{D}} = \|x_1 e + x_2 e^\dagger\|_{\mathbb{D}} = \|x_1\|_1 e + \|x_2\|_2 e^\dagger. \tag{2.3}$$

**Definition 2.2** Let  $\Omega \subset \mathbb{BC} \rightarrow \mathbb{BC}$  be a function defined on the domain  $\Omega$ . The derivative  $f'(z_0)$  of the function  $f$  at a point  $z_0 \in \Omega$  is the limit, if it exists,

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

such that  $z - z_0$  is an invertible bicomplex number. If  $f$  is derivable for all  $z \in \Omega$ , we say that it is a bicomplex holomorphic function in  $\Omega$ .

**Theorem 2.3** A bicomplex valued function  $f: \Omega \rightarrow \mathbb{BC}$  defined as

$$f(z) = f_1(z_1)e + f_2(z_2)e^\dagger$$

is a bicomplex holomorphic function on  $\Omega$  if and only if  $f_1$  and  $f_2$  are complex holomorphic functions on  $\Omega_1$  and  $\Omega_2$  respectively. Its derivatives is given by

$$f'(z) = f_1'(z_1)e + f_2'(z_2)e^\dagger.$$

The unit disk in  $\mathbb{C}(i)$  will be denoted by  $U$ . The bidisk  $\mathbb{U}$  in  $\mathbb{BC}$  is the product-type domain:

$$\mathbb{U} = U(e + e^\dagger) = U_1e + U_1e^\dagger = \{z_1e + z_2e^\dagger: z_1, z_2 \in U\}.$$

Note that such bidisk should be seen as the cartesian product  $U \times U$  in the idempotent  $\mathbb{C}^2(i)$ .

### 3 Bio-complex Bloch spaces

**Definition 3.1** <sup>[9]</sup> A bicomplex holomorphic function  $f$  in  $\mathbb{U}$  is said to be bicomplex Bloch space  $\mathcal{B}$  if

$$\|f\|_{\mathcal{B}} = \sup\{(1 - \|z\|_{\mathbb{D}}^2) \|f'(z)\|_{\mathbb{D}}: z \in \mathbb{U}\} < \infty.$$

The little bicomplex Bloch space  $\mathcal{B}_0$  is the closed subspace of  $\mathcal{B}$  consisting of functions  $f$  with :

$$\lim_{\|z\|_{\mathbb{D}} \rightarrow 1^-} (1 - \|z\|_{\mathbb{D}}^2) \|f'(z)\|_{\mathbb{D}} = 0.$$

The hyperbolic norm in the bicomplex Bloch space is defined as

$$\|f\|_{\mathbb{D}} = \|f(0)\|_{\mathbb{D}} + \|f\|_{\mathcal{B}}.$$

With this hyperbolic norm, the bicomplex Bloch space is complete.

By idempotent decomposition it is immediate that:

$$\mathcal{B} = \mathcal{B}_1e + \mathcal{B}_2e^\dagger \text{ and } \mathcal{B}_0 = \mathcal{B}_{0,1}e + \mathcal{B}_{0,2}e^\dagger, \text{ where } \mathcal{B}_l \text{ and } \mathcal{B}_{0,l} \text{ are the complex Bloch spaces, where } l = 1, 2.$$

**Lemma 3.2** <sup>[9]</sup> Let  $a \in \mathbb{U}$  and define the bicomplex Mobius transformation  $T: \mathbb{U} \rightarrow \mathbb{U}$  as:

$$T(z) = \Delta \frac{a - z}{1 - a^*z}, \text{ with } \|\Delta\|_{\mathbb{D}} = 1.$$

Then

$$(1 - \|z\|_{\mathbb{D}}^2) \|T'(z)\|_{\mathbb{D}} = 1 - \|T(z)\|_{\mathbb{D}}^2.$$

In particular  $\|T(z)\|_{\mathbb{D}} = 1$  if and only if  $\|z\|_{\mathbb{D}} = 1$ , that is, if  $z$  belongs to the distinguished boundary of  $\mathbb{U}$ .

**Lemma 3.3** <sup>[9]</sup> Let  $f: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic function. Then

$$\|f'(z)\|_{\mathbb{D}} \leq \frac{1 - \|f(z)\|_{\mathbb{D}}^2}{1 - \|z\|_{\mathbb{D}}^2}, \text{ for all } z \in \mathbb{U}.$$

The equality occurs if and only if  $f$  is a Mobius transformation of the bidisk.

Let  $f: \mathbb{U} \rightarrow \mathbb{BC}$  be a bicomplex holomorphic function and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . Thus, with  $z = z_1e + z_2e^\dagger \in \mathbb{U}$ , we can write  $f(z) = f_1(z_1)e + f_2(z_2)e^\dagger$ , where  $f_1$  and  $f_2$  are complex valued holomorphic functions on the unit disk  $U$  and  $\phi(z) = \phi_1(z_1)e + \phi_2(z_2)e^\dagger$ , where  $\phi_1$  and  $\phi_2$  are holomorphic self maps of the unit disk  $U$ .

**Definition 3.4** <sup>[4]</sup> Let  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . We define the composition operator  $C_\phi$  induced from  $\phi$  on  $\mathcal{B}$  by

$$C_\phi f = f \circ \phi, \forall f \in \mathcal{B}.$$

Further, from lemma [see <sup>[4]</sup>], we can write

$$C_{\phi}f(z) = C_{\phi_1}f_1(z_1)e + C_{\phi_2}f_2(z_2)e^{\dagger},$$

where  $C_{\phi_l}$ , for  $l = 1, 2$  is a composition operator on complex Bloch spaces.

**Definition 3.5** Let  $u$  be a fixed bicomplex holomorphic function on  $\mathbb{U}$  and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . Then we can define weighted composition operator  $u.C_{\phi}$  on  $\mathcal{B}$  by

$$u.C_{\phi}f = u.f \circ \phi, f \in \mathcal{B}.$$

We can write  $u.C_{\phi}f$  as

$$u.C_{\phi}f(z) = u_1C_{\phi_1}f_1(z_1)e + u_2C_{\phi_2}f_2(z_2)e^{\dagger},$$

and its derivatives is given by

$$\begin{aligned} [u(z).f \circ \phi(z)]' &= [u_1(z_1).f_1 \circ \phi_1(z_1)]'e + [u_2(z_2).f_2 \circ \phi_2(z_2)]'e^{\dagger} \\ &= (u_1'(z_1).f_1 \circ \phi_1(z_1) + u_1(z_1).f_1'(\phi_1(z_1)).\phi_1'(z_1))e + (u_2'(z_2).f_2 \circ \phi_2(z_2) + u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2))e^{\dagger} \\ &= [(u_1'(z_1).f_1 \circ \phi_1(z_1))e + (u_2'(z_2).f_2 \circ \phi_2(z_2))e^{\dagger}] + [(u_1(z_1).f_1'(\phi_1(z_1)).\phi_1'(z_1))e + (u_2(z_2).f_2'(\phi_2(z_2)).\phi_2'(z_2))e^{\dagger}] \\ &= (u'(z).f \circ \phi(z)) + (u(z).f'(\phi(z)).\phi'(z)). \end{aligned}$$

**Proposition 3.6** Let  $f: \mathbb{U} \rightarrow \mathbb{BC}$  be a bicomplex holomorphic function. If  $f \in \mathcal{B}$ , then

$$\|f(z) - f(w)\|_{\mathbb{D}} \leq \|f\|_{\mathcal{B}} \beta(z, w).$$

*Proof.* For any  $f \in \mathcal{B}$ , there exist two holomorphic functions  $f_1, f_2: \mathbb{U} \rightarrow \mathbb{C}(i)$  such that  $f(z) = f_1(z_1)e + f_2(z_2)e^{\dagger}$ . Since  $f \in \mathcal{B}$ , it follows that  $f_1$  and  $f_2$  lie in the complex Bloch spaces. So by using <sup>[7]</sup>, it follows that

$$\begin{aligned} \|f(z) - f(w)\|_{\mathbb{D}} &= \|(f_1(z_1) - f_1(w_1))e + (f_2(z_2) - f_2(w_2))e^{\dagger}\|_{\mathbb{D}} \\ &= \|f_1(z_1) - f_1(w_1)\|_1 e + \|f_2(z_2) - f_2(w_2)\|_2 e^{\dagger} \leq \|f_1\|_{\mathcal{B}_1} \beta_1(z_1, w_1)e + \|f_2\|_{\mathcal{B}_2} \beta_2(z_2, w_2)e^{\dagger} \\ &= (\|f_1\|_{\mathcal{B}_1} e + \|f_2\|_{\mathcal{B}_2} e^{\dagger})(\beta_1(z_1, w_1)e + \beta_2(z_2, w_2)e^{\dagger}) \\ &= \|f\|_{\mathcal{B}} \beta(z, w). \end{aligned}$$

Thus,  $\|f(z) - f(w)\|_{\mathbb{D}} \leq \|f\|_{\mathcal{B}} \beta(z, w)$ .

**Proposition 3.7** Let  $f: \mathbb{U} \rightarrow \mathbb{BC}$  be a bicomplex holomorphic function. If  $f \in \mathcal{B}$ , then

$$\|f(z)\|_{\mathbb{D}} \leq \frac{1}{\log 2} \|f\|_{\mathcal{B}} \log \frac{2}{1 - \|z\|_{\mathbb{D}}^2}. \quad (3.1)$$

*Proof.* For each  $f \in \mathcal{B}$  and  $z \in \mathbb{U}$ , we can write  $f(z)$  as

$$f(z) = f_1(z_1)e + f_2(z_2)e^{\dagger},$$

so by using <sup>[7]</sup>, it follows that

$$\begin{aligned} \|f(z)\|_{\mathbb{D}} &= \|f_1(z_1)\|_1 e + \|f_2(z_2)\|_2 e^{\dagger} \leq \frac{1}{\log 2} \|f_1\|_{\mathcal{B}_1} \log \frac{2}{1 - \|z_1\|_1^2} e + \frac{1}{\log 2} \|f_2\|_{\mathcal{B}_2} \log \frac{2}{1 - \|z_2\|_2^2} e^{\dagger} \\ &= \frac{1}{\log 2} (\|f_1\|_{\mathcal{B}_1} e + \|f_2\|_{\mathcal{B}_2} e^{\dagger}) (\log \frac{2}{1 - \|z_1\|_1^2} e + \log \frac{2}{1 - \|z_2\|_2^2} e^{\dagger}) \\ &= \frac{1}{\log 2} \|f\|_{\mathcal{B}} \log \frac{2}{1 - \|z\|_{\mathbb{D}}^2}. \end{aligned}$$

**Proposition 3.8** Let  $\mathcal{B}$  be a bicomplex Bloch space and let  $f: \mathbb{U} \rightarrow \mathbb{BC}$  be a bicomplex holomorphic function. If  $f \in \mathcal{B}$ , then  $f \circ \phi \in \mathcal{B}$ .

*Proof.* If  $f \in \mathcal{B}$ , then  $(1 - \|z\|_{\mathbb{D}}^2) \|f'(z)\|_{\mathbb{D}} < \infty$ . Now,

$$\|(f \circ \phi)'(z)\|_{\mathbb{D}} = \|f'(\phi(z)) \cdot \phi'(z)\|_{\mathbb{D}} = \|f'(\phi(z))\|_{\mathbb{D}} \cdot \|\phi'(z)\|_{\mathbb{D}}$$

$$(1 - \|z\|_{\mathbb{D}}^2) \|(f \circ \phi)'(z)\|_{\mathbb{D}} = (1 - \|z\|_{\mathbb{D}}^2) \|f'(\phi(z))\|_{\mathbb{D}} \|\phi'(z)\|_{\mathbb{D}}.$$

Since,  $\phi(z)$  is bicomplex mobius transformation on  $\mathbb{U}$ , so

$$(1 - \|z\|_{\mathbb{D}}^2) \|\phi'(z)\|_{\mathbb{D}} = (1 - \|\phi(z)\|_{\mathbb{D}}^2).$$

$$(1 - \|z\|_{\mathbb{D}}^2) \|(f \circ \phi)'(z)\|_{\mathbb{D}} = (1 - \|\phi(z)\|_{\mathbb{D}}^2) \|f'(\phi(z))\|_{\mathbb{D}}$$

$$= \|f\|_{\mathcal{B}} < \infty.$$

Thus,  $f \circ \phi \in \mathcal{B}$ .

**Theorem 3.9** Let  $\mathcal{B}$  be a bicomplex Bloch space and let  $u.C_{\phi}: \mathcal{B} \rightarrow \mathcal{B}$  be a weighted composition operator on  $\mathcal{B}$ . Then  $u.C_{\phi}$  is bounded on  $\mathcal{B}$  if and only if  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .

*Proof.* Suppose that  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . We can write  $u.C_{\phi}$  as

$$u.C_{\phi} = u_1.C_{\phi_1}e + u_2.C_{\phi_2}e^{\dagger}.$$

Since, both  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . This implies that  $\|u_1.C_{\phi_1}\|_{\mathcal{B}_1} < \infty$  and  $\|u_2.C_{\phi_2}\|_{\mathcal{B}_2} < \infty$ . Then we have

$$\|u.C_{\phi}\|_{\mathcal{B}} = \|u_1.C_{\phi_1}\|_{\mathcal{B}_1}e + \|u_2.C_{\phi_2}\|_{\mathcal{B}_2}e^{\dagger}$$

$$< \infty.$$

Thus,  $u.C_{\phi}$  is bounded on bicomplex Bloch space  $\mathcal{B}$ .

Conversely, suppose that  $u.C_{\phi}$  is bounded on bicomplex Bloch space  $\mathcal{B}$ . Now, we have to show that  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$  respectively. There exist some constants  $M_1$  and  $M_2$  such that  $M = M_1e + M_2e^{\dagger}$ . Then we have

$$\|u.C_{\phi}\|_{\mathcal{B}} < M = M_1e + M_2e^{\dagger}.$$

Further,  $\|u.C_{\phi}\|_{\mathcal{B}} = \|u_1.C_{\phi_1}\|_{\mathcal{B}_1}e + \|u_2.C_{\phi_2}\|_{\mathcal{B}_2}e^{\dagger}$ . Then

$$\|u_1.C_{\phi_1}\|_{\mathcal{B}_1}e + \|u_2.C_{\phi_2}\|_{\mathcal{B}_2}e^{\dagger} < M_1e + M_2e^{\dagger}.$$

Therefore,  $\|u_1.C_{\phi_1}\|_{\mathcal{B}_1} < M_1$  and  $\|u_2.C_{\phi_2}\|_{\mathcal{B}_2} < M_2$ . Hence " $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ ."

#### 4. Conditions for boundedness and compactness

**Theorem 4.1** Let  $u$  be a bicomplex holomorphic function on  $\mathbb{U}$  and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . Then  $u.C_{\phi}$  is bounded on a bicomplex Bloch space  $\mathcal{B}$  if and only if the following are satisfied:

$$1. \sup_{z \in \mathbb{U}} (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \log \frac{2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} < \infty.$$

$$2. \sup_{z \in \mathbb{U}} \frac{1 - \|z\|_{\mathbb{D}}^2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} \|u(z)\phi'(z)\|_{\mathbb{D}} < \infty.$$

*Proof.* Suppose  $u.C_{\phi}$  is bounded on a bicomplex Bloch space  $\mathcal{B}$ . By above theorem,  $u.C_{\phi}$  is bounded on bicomplex Bloch space if and only if  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are bounded on complex Bloch spaces. We can decompose the L.H.S of (i) and (ii) part as:

$$\sup_{z \in \mathbb{U}} (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \log \frac{2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} = \sup_{z_1 \in U} (1 - \|z_1\|_1^2) \|u_1'(z_1)\|_1 \log \frac{2}{1 - \|\phi_1(z_1)\|_1^2} e + \sup_{z_2 \in U} (1 - \|z_2\|_2^2) \|u_2'(z_2)\|_2 \log \frac{2}{1 - \|\phi_2(z_2)\|_2^2} e^{\dagger}$$

$$\sup_{z \in \mathbb{U}} \frac{1 - \|z\|_{\mathbb{D}}^2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} \|u(z)\phi'(z)\|_{\mathbb{D}} = \sup_{z_1 \in U} \frac{1 - \|z_1\|_1^2}{1 - \|\phi_1(z_1)\|_1^2} \|u_1(z_1)\phi_1'(z_1)\|_1 e + \sup_{z_2 \in U} \frac{1 - \|z_2\|_2^2}{1 - \|\phi_2(z_2)\|_2^2} \|u_2(z_2)\phi_2'(z_2)\|_2 e^{\dagger}.$$

By [7], theorem 1],  $\sup_{z_l \in U} (1 - \|z_l\|_l^2) \|u_l'(z_l)\|_l \log \frac{2}{1 - \|\phi_l(z_l)\|_l^2} < \infty$  and

$\sup_{z_l \in U} \frac{1 - \|z_l\|_l^2}{1 - \|\phi_l(z_l)\|_l^2} \|u_l(z_l)\phi_l'(z_l)\|_l < \infty$  if and only if  $u_l C_{\phi_l}$  is bounded on complex Bloch spaces, where  $l = 1, 2$ .

Thus, it follows that

$$\sup_{z \in \mathbb{U}} \frac{1 - \|z\|_{\mathbb{D}}^2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} \|u(z)\phi'(z)\|_{\mathbb{D}} < \infty \text{ and}$$

$$\sup_{z \in \mathbb{U}} (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \log \frac{2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} < \infty.$$

Conversely, suppose that (i) and (ii) conditions are hold. We have to show that  $u.C_{\phi}$  is bounded on bicomplex Bloch space  $\mathcal{B}$ . Further,

$$\begin{aligned} \|u.C_{\phi}f\|_{\mathcal{B}} &= (1 - \|z\|_{\mathbb{D}}^2) \|(u.C_{\phi}f)'(z)\|_{\mathbb{D}} \\ &= (1 - \|z\|_{\mathbb{D}}^2) \|(u.f \circ \phi)'(z)\|_{\mathbb{D}} \\ &= (1 - \|z_1\|_1^2) \|(u_1.f_1 \circ \phi_1)'(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|(u_2.f_2 \circ \phi_2)'(z_2)\|_2 e^{\dagger} \\ &= (1 - \|z_1\|_1^2) \|u_1'(z_1).f_1 \circ \phi_1(z_1) + u_1(z_1).f_1'(\phi_1(z_1))(\phi_1'(z_1))\|_1 e + (1 - \|z_2\|_2^2) \\ &\quad \|u_2'(z_2).f_2 \circ \phi_2(z_2) + u_2(z_2).f_2'(\phi_2(z_2))(\phi_2'(z_2))\|_2 e^{\dagger} \\ &\leq (1 - \|z_1\|_1^2) [\|u_1'(z_1)\|_1 \cdot \|f_1 \circ \phi_1(z_1)\|_1 + \|u_1(z_1)(\phi_1'(z_1))\|_1 \cdot \|f_1'(\phi_1(z_1))\|_1] e + (1 - \|z_2\|_2^2) [\|u_2'(z_2)\|_2 \cdot \\ &\quad \|f_2 \circ \phi_2(z_2)\|_2 + \|u_2(z_2)(\phi_2'(z_2))\|_2 \cdot \|f_2'(\phi_2(z_2))\|_2] e^{\dagger} \\ &= (1 - \|z_1\|_1^2) \|u_1'(z_1)\|_1 \cdot \|f_1 \circ \phi_1(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|u_2'(z_2)\|_2 \cdot \|f_2 \circ \phi_2(z_2)\|_2 e^{\dagger} \\ &\quad + (1 - \|z_1\|_1^2) \|u_1(z_1)(\phi_1'(z_1))\|_1 \cdot \|f_1'(\phi_1(z_1))\|_1 e + (1 - \|z_2\|_2^2) \|u_2(z_2)(\phi_2'(z_2))\|_2 \cdot \\ &\quad \|f_2'(\phi_2(z_2))\|_2 e^{\dagger} \\ &= (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \cdot \|f \circ \phi(z)\|_{\mathbb{D}} + (1 - \|z\|_{\mathbb{D}}^2) \|u(z)(\phi'(z))\|_{\mathbb{D}} \cdot \|f'(\phi(z))\|_{\mathbb{D}} \\ &= (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \cdot \|f \circ \phi(z)\|_{\mathbb{D}} + \frac{(1 - \|z\|_{\mathbb{D}}^2)}{(1 - \|\phi(z)\|_{\mathbb{D}}^2)} (1 - \|\phi(z)\|_{\mathbb{D}}^2) \|f'(\phi(z))\|_{\mathbb{D}} \cdot \\ &\quad \|u(z)(\phi'(z))\|_{\mathbb{D}} \\ &\leq (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \frac{1}{\log 2} \|f\|_{\mathcal{B}} \log \frac{2}{1 - \|z\|_{\mathbb{D}}^2} + \frac{(1 - \|z\|_{\mathbb{D}}^2)}{(1 - \|\phi(z)\|_{\mathbb{D}}^2)} \|f\|_{\mathcal{B}} \\ &\quad \|u(z)(\phi'(z))\|_{\mathbb{D}}, \end{aligned}$$

where we use (3.1) in last inequality. So, (i) and (ii) condition implies that R.H.S is bounded by some constant times  $\|f\|_{\mathcal{B}}$ . Thus,  $u.C_{\phi}$  is bounded on a bicomplex Bloch space  $\mathcal{B}$ .

**Theorem 4.2** Let  $u$  be a bicomplex holomorphic function on  $\mathbb{U}$  and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . Then  $u.C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$  if and only if  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are compact on a bicomplex Bloch space  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .

*Proof.* Suppose that  $u.C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$ . Let  $\{f_n\}$  be a bounded sequence in  $\mathcal{B}$ , each  $\{f_n\}$  has complex idempotent decomposition  $f_n = f_{n,1}e + f_{n,2}e^{\dagger}$ . Then  $f_{n,1}$  and  $f_{n,2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . Since,  $u.C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$ , then  $\|u.C_{\phi}f_n\|_{\mathcal{B}} \rightarrow 0$  as  $n \rightarrow \infty$ .

We have  $\|u.C_{\phi}f_n\|_{\mathcal{B}} = \|u_1.C_{\phi_1}f_{n,1}\|_{\mathcal{B}_1} e + \|u_2.C_{\phi_2}f_{n,2}\|_{\mathcal{B}_2} e^{\dagger}$  which means as  $n \rightarrow \infty$ ,  $\|u_1.C_{\phi_1}f_{n,1}\|_{\mathcal{B}_1} \rightarrow 0$  and  $\|u_2.C_{\phi_2}f_{n,2}\|_{\mathcal{B}_2} \rightarrow 0$ . Thus, both  $u_1.C_{\phi_1}$  and  $u_2.C_{\phi_2}$  are compact on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ .

Conversely, suppose that  $u_1 \cdot C_{\phi_1}$  and  $u_2 \cdot C_{\phi_2}$  are compact on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . So,  $\|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} \rightarrow 0$  and  $\|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} \rightarrow 0$ , as  $n \rightarrow \infty$ .

Now, we have  $\|u \cdot C_{\phi} f_n\|_{\mathcal{B}} = \|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} e + \|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} e^{\dagger}$ . Thus, as  $n \rightarrow \infty$ ,  $\|u \cdot C_{\phi} f_n\|_{\mathcal{B}} \rightarrow 0$ .

**Theorem 4.3** Let  $u$  be a bicomplex holomorphic function on  $\mathbb{U}$  and  $\phi: \mathbb{U} \rightarrow \mathbb{U}$  be a bicomplex holomorphic self map of  $\mathbb{U}$ . Suppose that  $u \cdot C_{\phi}$  is bounded on a bicomplex Bloch space  $\mathcal{B}$ . Then  $u \cdot C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$  if and only if the following are satisfied.

$$1. \lim_{\|\phi(z)\|_{\mathbb{D}} \rightarrow 1^-} (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \log \frac{2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} = 0$$

$$2. \lim_{\|\phi(z)\|_{\mathbb{D}} \rightarrow 1^-} \frac{1 - \|z\|_{\mathbb{D}}^2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} \|u(z)\phi'(z)\|_{\mathbb{D}} = 0$$

*Proof.* Suppose that  $u \cdot C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$ . Let  $\{f_n\}$  be a bounded sequence in  $\mathcal{B}$ , each  $\{f_n\}$  has complex idempotent decomposition  $f_n = f_{n,1}e + f_{n,2}e^{\dagger}$ . Then  $f_{n,1}$  and  $f_{n,2}$  are bounded on  $\mathcal{B}_1$  and  $\mathcal{B}_2$ . Since,  $u \cdot C_{\phi}$  is compact on a bicomplex Bloch space  $\mathcal{B}$ , we have

$$\|u \cdot C_{\phi} f_n\|_{\mathcal{B}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Further,  $\|u \cdot C_{\phi} f_n\|_{\mathcal{B}} = \|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} e + \|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} e^{\dagger}$ . Then, by above theorem,  $\|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} \rightarrow 0$  and  $\|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} e^{\dagger} \rightarrow 0$ , as  $n \rightarrow \infty$ . This implies that  $u_1 \cdot C_{\phi_1}$  and  $u_2 \cdot C_{\phi_2}$  are compact on complex Bloch spaces  $\mathcal{B}_1$  and  $\mathcal{B}_2$  respectively. Now,

$$\begin{aligned} \|u \cdot C_{\phi} f_n\|_{\mathcal{B}} &= \|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} e + \|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} e^{\dagger} \\ &= (1 - \|z_1\|_1^2) \|(u_1 \cdot f_{n,1} \circ \phi_1)'(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|(u_2 \cdot f_{n,2} \circ \phi_2)'(z_2)\|_2 e^{\dagger} \\ &= [(1 - \|z_1\|_1^2) \|u_1'(z_1) \cdot f_1 \circ \phi_1(z_1) + u_1(z_1) \cdot f_1'(\phi_1(z_1)) \cdot \phi_1'(z_1)\|_1] e \\ &\quad + [(1 - \|z_2\|_2^2) \|u_2'(z_2) \cdot f_2 \circ \phi_2(z_2) + u_2(z_2) \cdot f_2'(\phi_2(z_2)) \cdot \phi_2'(z_2)\|_2] e^{\dagger} \\ &\leq [(1 - \|z_1\|_1^2) \|u_1'(z_1) \cdot f_1 \circ \phi_1(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|u_2'(z_2) \cdot f_2 \circ \phi_2(z_2)\|_2 e^{\dagger}] \\ &\quad + [(1 - \|z_1\|_1^2) \|u_1(z_1) \cdot f_1'(\phi_1(z_1)) \cdot \phi_1'(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|u_2(z_2) \cdot f_2'(\phi_2(z_2)) \cdot \phi_2'(z_2)\|_2 e^{\dagger}] \\ &\leq (1 - \|z_1\|_1^2) \|u_1'(z_1)\|_1 \cdot \frac{1}{\log 2} \|f_1\|_{\mathcal{B}_1} \log \frac{2}{(1 - \|z_1\|_1^2)} e \\ &\quad + (1 - \|z_2\|_2^2) \|u_2'(z_2)\|_2 \cdot \frac{1}{\log 2} \|f_2\|_{\mathcal{B}_2} \log \frac{2}{(1 - \|z_2\|_2^2)} e^{\dagger} \\ &\quad + \frac{(1 - \|z_1\|_1^2)}{(1 - \|\phi(z_1)\|_1^2)} (1 - \|\phi(z_1)\|_1^2) \|f_1'(\phi_1(z_1))\|_1 \|u_1(z_1) \phi_1'(z_1)\|_1 e \\ &\quad + \frac{(1 - \|z_2\|_2^2)}{(1 - \|\phi(z_2)\|_2^2)} (1 - \|\phi(z_2)\|_2^2) \|f_2'(\phi_2(z_2))\|_2 \|u_2(z_2) \phi_2'(z_2)\|_2 e^{\dagger} \\ &= (1 - \|z_1\|_1^2) \|u_1'(z_1)\|_1 \cdot \frac{1}{\log 2} \|f_1\|_{\mathcal{B}_1} \log \frac{2}{(1 - \|z_1\|_1^2)} e \\ &\quad + (1 - \|z_2\|_2^2) \|u_2'(z_2)\|_2 \cdot \frac{1}{\log 2} \|f_2\|_{\mathcal{B}_2} \log \frac{2}{(1 - \|z_2\|_2^2)} e^{\dagger} \\ &\quad + \frac{(1 - \|z_1\|_1^2)}{(1 - \|\phi(z_1)\|_1^2)} \|f_1\|_1 \|u_1(z_1) \phi_1'(z_1)\|_1 e \\ &\quad + \frac{(1 - \|z_2\|_2^2)}{(1 - \|\phi(z_2)\|_2^2)} \|f_2\|_2 \|u_2(z_2) \phi_2'(z_2)\|_2 e^{\dagger}. \end{aligned}$$

Therefore, from [7], we have  $\lim_{\|\phi_l(z)\|_l \rightarrow 1^-} (1 - \|z_l\|_l^2) u_l'(z_l) \frac{1}{\log 2} \|f_l\|_l \log \frac{2}{(1 - \|z_l\|_l^2)} = 0$  and  $\lim_{\|\phi_l(z)\|_l \rightarrow 1^-} \frac{(1 - \|z_l\|_l^2)}{(1 - \|\phi_l(z)\|_l^2)} \|f_l\|_l \|u_l(z_l) \phi_l'(z_l)\|_l = 0$

$\|f_l\|_l(u_l(z_l)\phi_l'(z_l)) = 0$ , where  $l = 1, 2$ .

Thus, it follows that

$$\lim_{|\phi(z)| \rightarrow 1^-} (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \log \frac{2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} = 0 \text{ and}$$

$$\lim_{|\phi(z)| \rightarrow 1^-} \frac{1 - \|z\|_{\mathbb{D}}^2}{1 - \|\phi(z)\|_{\mathbb{D}}^2} \|u(z)\phi'(z)\|_{\mathbb{D}} = 0.$$

Conversely suppose that (i) and (ii) conditions are holds. Now, we have

$$\begin{aligned} \|u \cdot C_{\phi} f_n\|_{\mathcal{B}} &= \|u_1 \cdot C_{\phi_1} f_{n,1}\|_{\mathcal{B}_1} e + \|u_2 \cdot C_{\phi_2} f_{n,2}\|_{\mathcal{B}_2} e^{\dagger} \\ &= \|u_1 \cdot f_{n,1} \circ \phi_1\|_{\mathcal{B}_1} e + \|u_2 \cdot f_{n,2} \circ \phi_2\|_{\mathcal{B}_2} e^{\dagger} \\ &= (1 - \|z_1\|_1^2) \| [u_1(z_1) \cdot f_{n,1} \circ \phi_1(z_1)]' \|_1 e + (1 - \|z_2\|_2^2) \| [u_2(z_2) \cdot f_{n,2} \circ \phi_2(z_2)]' \|_2 e^{\dagger} \\ &= [(1 - \|z_1\|_1^2) \|u_1'(z_1) \cdot f_{n,1} \circ \phi_1(z_1) + u_1(z_1) \cdot f_{n,1}'(\phi_1(z_1)) \cdot \phi_1'(z_1)\|_1] e \\ &\quad + [(1 - \|z_2\|_2^2) \|u_2'(z_2) \cdot f_{n,2} \circ \phi_2(z_2) + u_2(z_2) \cdot f_{n,2}'(\phi_2(z_2)) \cdot \phi_2'(z_2)\|_2] e^{\dagger} \\ &\leq [(1 - \|z_1\|_1^2) \|u_1'(z_1) \cdot f_{n,1} \circ \phi_1(z_1)\|_1 e + (1 - \|z_2\|_2^2) \|u_2'(z_2) \cdot f_{n,2} \circ \phi_2(z_2)\|_2 e^{\dagger}] \\ &\quad + [(1 - \|z_1\|_1^2) \|u_1(z_1) \cdot f_{n,1}'(\phi_1(z_1)) \cdot \phi_1'(z_1)\|_1 e \\ &\quad + (1 - \|z_2\|_2^2) \|u_2(z_2) \cdot f_{n,2}'(\phi_2(z_2)) \cdot \phi_2'(z_2)\|_2 e^{\dagger}] \\ &= [(1 - \|z\|_{\mathbb{D}}^2) \|u'(z) \cdot f_n \circ \phi(z)\|_{\mathbb{D}}] + [(1 - \|z\|_{\mathbb{D}}^2) \|u(z) \cdot f_n'(\phi(z)) \cdot \phi'(z)\|_{\mathbb{D}}] \\ &\leq (1 - \|z\|_{\mathbb{D}}^2) \|u'(z)\|_{\mathbb{D}} \cdot \frac{1}{\log 2} \|f_n\|_{\mathcal{B}} \log \frac{2}{(1 - \|z\|_{\mathbb{D}}^2)} \\ &\quad + \frac{(1 - \|z\|_{\mathbb{D}}^2)}{(1 - \|\phi(z)\|_{\mathbb{D}}^2)} \|f_n\|_{\mathcal{B}} \|u(z) \cdot \phi'(z)\|_{\mathbb{D}}. \end{aligned}$$

Since, conditions (i) and (ii) holds. So,  $\|u \cdot C_{\phi} f_n\|_{\mathcal{B}} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $u \cdot C_{\phi}$  is a compact on bicomplex Bloch space  $\mathcal{B}$ .

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