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A new extended weighted distribution with comprehensive statistical properties, estimations and its applications

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Abstract

In this study, we have introduced a new distribution by applying the weighted version to baseline X-Rani distribution termed as weighted XRani distribution. The formulated new distribution has been encompassed with several statistical properties via order statistics, moments and reliability measures, etc. In addition its parameters are estimated by applying the maximum likelihood estimation. Further a real life time data set has been investigated and fitted to examine the dominance and capability of a new distribution.

Keywords: Weighted distribution, X-Rani distribution, order statistics, reliability measures, maximum likelihood estimation

1. Introduction

Even though the distribution theory retains more popularity and significance a situation has been emerged to introduce a shape parameter to the existing classical distribution because the inclusion of shape parameter brings a greater flexibility and compressibility to existing classical distribution. This shape parameter can be introduced through several techniques. One such method is of weighted technique. In statistics, the concept of weighted distribution is too important and useful because it provides a significant technique to extend existing classical distribution by incorporating an extra parameter. Fisher (1934) ^[4] suggested the idea of weighted distribution to describe how method of ascertainment biases should impact the formulation of distribution of recorded observations which subsequently developed by Rao (1965) ^[11] in an integrated way for problems where the observations fall in non-experimental, non-replicated and non-random manner. The weighted distributions were developed to attain the observations in a comprehensive and collective manner. The weighted distribution is found to be specified and important aspect in statistics, probability and mathematics. The weighted distributions also have played a key role because of several experimental problems exists in biased samples by nature. The weighted distribution is considered as a fundamental tool in selection of appropriate models for observed data obviously if samples are drawn without a proper frame. The weighted distributions captured a remarkable contribution for modeling the statistical data efficiently especially if the classical distributions are unsuitable. The weighted distributions introduced a fundamental approach to deal with interpretation of model stipulation and data representation problems. The weighted distributions are obviously applied even to modulate the cluster sampling, heterogeneity and extraneous variation in the data set and is also described that weighted distributions occur subsequently because of stochastic nature of biased data. The weighted distributions also provide a collective approach to deal with model biased data and also play a significant role in modeling, investigating and analyzing complicated lifetime data sets from several applied and engineering sciences like medicine, behavioral sciences, insurance and finance, etc.

A comprehensive and magnificent contribution has been done by various researchers to introduce several weighted probability distributions besides with their characterizations and applications. Shanker and Prodhani (2024) ^[16] proposed on weighted Amarendra distribution with properties and applications.

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Patil and Rao (1978) ^[9] developed weighted distributions and size biased sampling with applications to wild life populations and human families. Wani and Shafi (2021) ^[18] described weighted Lindley- quasi Xgamma distribution. Jan, Fatima and Ahmad (2017) ^[7] discussed on weighted Ailamujia distribution and its applications. Reshi *et al.* (2014) ^[12] suggested characterizations and estimations of size biased generalized Rayleigh distribution. Saghir and Khadim (2016) ^[14] derived and obtained the mathematical properties of length biased weighted Maxwell distribution. Ganaie and Rajagopalan (2021) ^[5] proposed the length biased weighted new quasi Lindley distribution and obtain its statistical properties and applications. Shamran (2020) presented modified weighted Pareto distribution type I with properties. Hassan *et al.* (2019) ^[6] introduced weighted quasi exponential distribution with properties and applications. Rajagopalan, Ganaie and Nair (2023) ^[10] observed and studied the weighted three parameter Pranav distribution and discussed briefly its properties and applications. Chaito *et al.* (2022) ^[2] obtained length biased gamma-Rayleigh distribution with applications. Ahmed *et al.* (2014) ^[12] developed structural properties of size biased gamma distribution.

XRani distribution is a single parametric distribution just like as the Rani distribution and the XRani distribution was established by Etaga *et al.* (2023) ^[3]. Its different mathematical properties like stochastic ordering, moments, survival function, hazard rate function, bonferroni and Lorenz curve, mean residual life function, Renyi entropy, odd function and stress-strength reliability has been described. For estimating its parameter, the method of maximum likelihood estimation has been discussed. Rani distribution is also one parametric lifetime distribution introduced by Shanker (2017) ^[17] established its numerous statistical properties and estimate its parameters through by method of maximum likelihood estimation and method of moments.

2. Weighted XRani Distribution

The probability density function of XRani distribution is given by

$$f(x; \theta) = \frac{\theta^5}{(\theta^5 + 24)^2} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x}; \quad x > 0, \theta > 0 \quad (1)$$

The cumulative distribution function of XRani distribution is given by

$$F(x; \theta) = 1 - \left\{ 1 + \frac{1}{(\theta^5 + 24)^2} \left(24\theta^4 x^4 + 96\theta^3 x^3 + 288\theta^2 x^2 + 576\theta x \right) \right\} e^{-\theta x}; \quad x > 0, \theta > 0 \quad (2)$$

Consider the random variable X represents non-negativity condition with probability density function $f(x)$. Let its non-negative weight function be $w(x)$ then probability density function of weighted random variable X_w is given by

$$f_w(x) = \frac{w(x) f(x)}{E(w(x))}, \quad x > 0$$

Where the non - negative weight function be $w(x)$ and $E(w(x)) = \int w(x) f(x) dx < \infty$.

In this paper, we have contemplated or constructed the weight function as $w(x) = x^c$ to get the weighted version of XRani distribution, then its probability density function is given by

$$f_w(x; \theta, c) = \frac{x^c f(x; \theta)}{E(x^c)} \quad (3)$$

Where $E(x^c) = \int_0^{\infty} x^c f(x; \theta) dx$

$$E(x^c) = \int_0^{\infty} x^c \frac{\theta^5}{(\theta^5 + 24)^2} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx$$

After simplification, we get

$$E(x^c) = \frac{\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5)}{\theta^c (\theta^5 + 24)^2} \quad (4)$$

After applying equation (1) and equation (4) in equation (3), we will get required probability density function of weighted XRani distribution given by

$$f_w(x; \theta, c) = \frac{x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} \quad (5)$$

The cumulative distribution function of weighted XRani distribution will be determined as

$$F_w(x; \theta, c) = \int_0^x f_w(x; \theta, c) dx$$

$$F_w(x; \theta, c) = \int_0^x \frac{x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx$$

$$F_w(x; \theta, c) = \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \int_0^x x^c \theta^{c+5} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx$$

$$F_w(x; \theta, c) = \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^c + 11 \int_0^x x^c e^{-\theta x} dx + 48\theta^c + 6 \int_0^x x^c e^{-\theta x} dx + 24\theta^c + 5 \int_0^x x^{c+4} e^{-\theta x} dx \right) \quad (6)$$

$$\text{By Putting } \theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$$

$$\text{Now } x = \frac{t}{\theta}$$

$$\text{When } x \rightarrow 0, t \rightarrow \theta x \text{ and when } x \rightarrow 0, t \rightarrow 0$$

After simplifying above equation (6), we will determine the cumulative distribution function of weighted XRani distribution which is given by

$$F_w(x; \theta, c) = \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right) \quad (7)$$

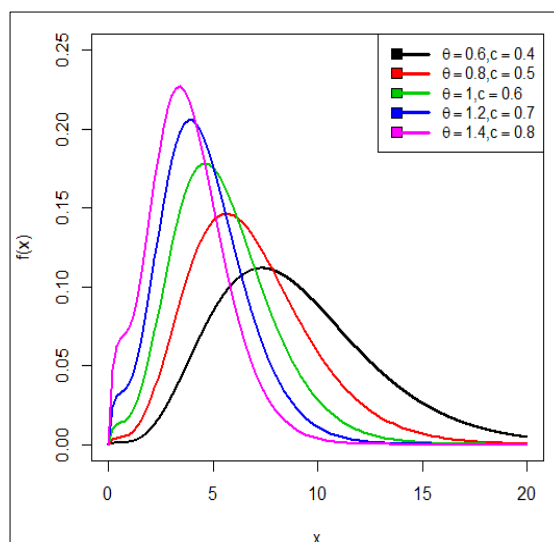


Fig 1: PDF plot weighted X Rani distribution

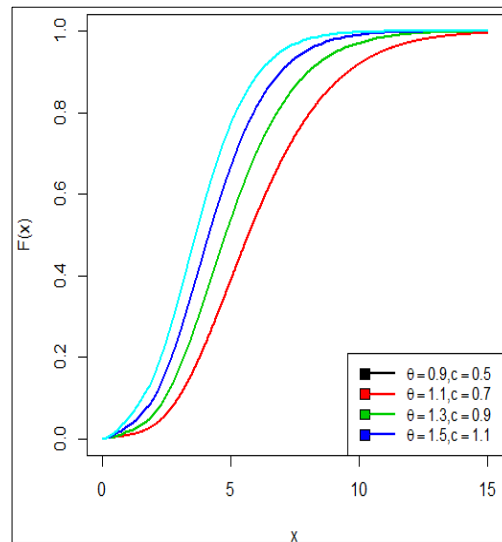


Fig 2: CDF plot of weighted X Rani distribution

3. Reliability Measures

In this portion, we have derived and described the reliability function, hazard rate function, reverse hazard rate function and Mills ratio of the weighted X-Rani distribution.

3.1 Reliability function

The reliability function of weighted XRani distribution is given by

$$R(x) = 1 - F_w(x; \theta, c)$$

$$R(x) = 1 - \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right)$$

3.2 Hazard function

The hazard rate function of proposed distribution is given by

$$h(x) = \frac{f_w(x; \theta, c)}{1 - F_w(x; \theta, c)}$$

$$h(x) = \frac{x^c \theta^{c+5} (\theta^6 + 48\theta + 24x^4) e^{-\theta x}}{(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5)) - (\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x))}$$

3.3 Reverse hazard function

The reverse hazard function is given by

$$h_r(x) = \frac{f_w(x; \theta, c)}{F_w(x; \theta, c)}$$

$$h_r(x) = \frac{x^c \theta^{c+5} (\theta^6 + 48\theta + 24x^4) e^{-\theta x}}{(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x))}$$

3.4 Mills Ratio

$$M.R = \frac{1}{h_r(x)}$$

$$M.R = \frac{(\theta^{10}\gamma(c+1, \theta x) + 48\theta^5\gamma(c+1, \theta x) + 24\gamma(c+5, \theta x))}{x^c \theta^{c+5} (\theta^6 + 48\theta + 24x^4) e^{-\theta x}}$$

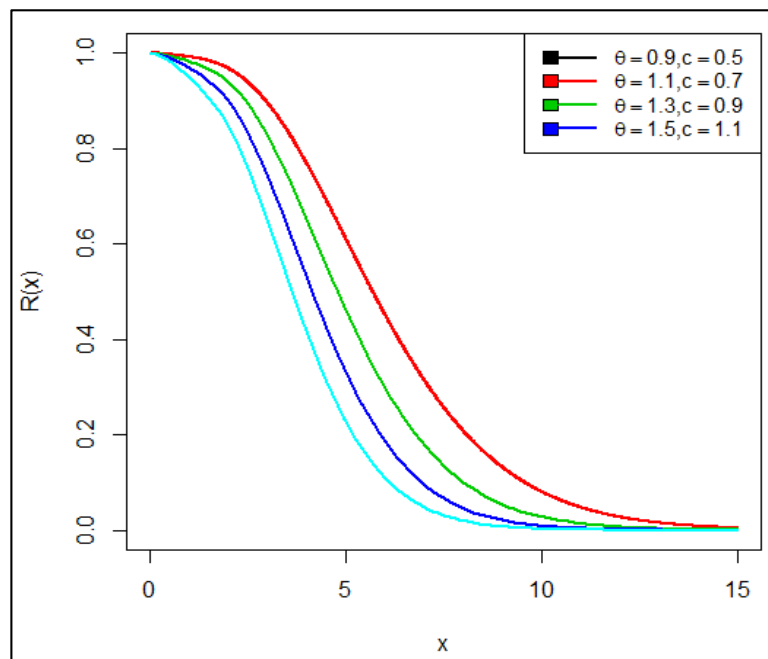


Fig 3: Reliability plot of weighted X rani distribution

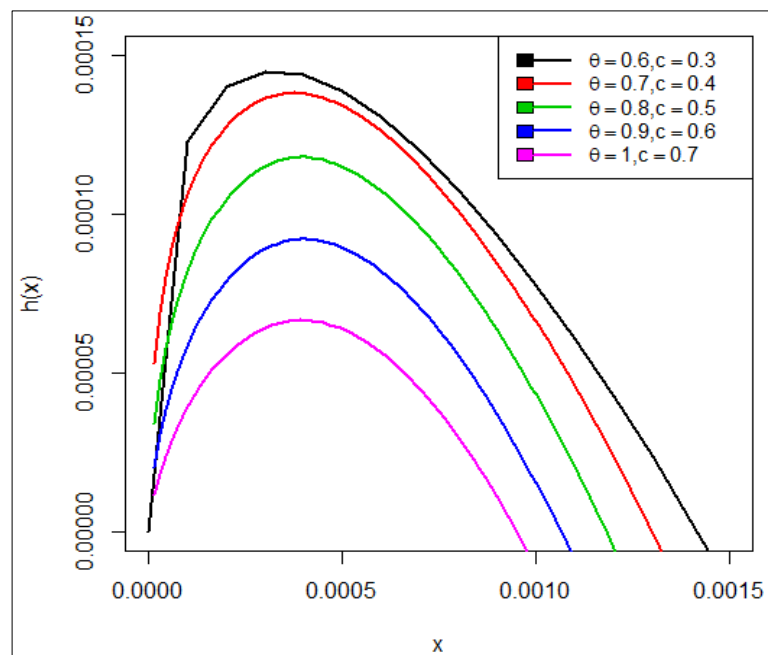


Fig 3: Hazard plot of weighted X rani distribution

4. Order Statistics

Order statistics is concerned with the result of an experiment by measuring the values of a random variable arranged in order of magnitude are termed as order statistics. Consider $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous population with probability density function $f_X(x)$ and cumulative distribution function $F_X(x)$, then the probability density function of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) (F_X(x))^{r-1} (1-F_X(x))^{n-r} \quad (8)$$

After applying equation (5) and equation (7) in equation (8), we will get the probability density function of r^{th} order statistics $X_{(r)}$ of weighted XRani distribution given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} \right) \\ \times \left(\frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right) \right)^{r-1} \\ \times \left(1 - \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right) \right)^{n-r}$$

For $r = n$, we will determine the probability density function of higher order statistic $X_{(n)}$ of weighted XRani distribution as

$$f_{x(n)}(x) = \frac{n x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} \\ \times \left(\frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right) \right)^{n-1}$$

For $r = 1$, we will determine the probability density function of first order statistic $X_{(1)}$ of weighted XRani distribution as

$$f_{x(1)}(x) = \frac{n x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} \\ \times \left(1 - \frac{1}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^{10} \gamma(c+1, \theta x) + 48\theta^5 \gamma(c+1, \theta x) + 24\gamma(c+5, \theta x) \right) \right)^{n-1}$$

5. Statistical Properties

In this portion, we have explored the different structural properties of weighted XRani distribution particularly its moments, harmonic mean and moment generating function.

5.1 Moments

Suppose random variable X represents the weighted XRani distribution with parameters θ and c , then the r^{th} order moment of weighted XRani distribution can be determined as

$$E(X^r) = \mu_r' = \int_0^{\infty} x^r f_w(x; \theta, c) dx$$

$$E(X^r) = \mu_r' = \int_0^{\infty} x^r \frac{x^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx$$

$$\begin{aligned}
E(X^r) &= \mu_r' = \int_0^\infty \frac{x^{c+r} \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx \\
E(X^r) &= \mu_r' = \frac{\theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \int_0^\infty x^{c+r} \left(\theta^6 + 48\theta + 24x^4 \right) e^{-\theta x} dx \\
E(X^r) &= \mu_r' = \frac{\theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 \int_0^\infty x^{(c+r+1)-1} e^{-\theta x} dx + 48\theta \int_0^\infty x^{(c+r+1)-1} e^{-\theta x} dx \right. \\
&\quad \left. + 24 \int_0^\infty x^{(c+r+5)-1} e^{-\theta x} dx \right) \quad (9)
\end{aligned}$$

After the simplification of equation (9), we obtain

$$E(X^r) = \mu_r' = \frac{\theta^{10} \Gamma(c+r+1) + 48\theta^5 \Gamma(c+r+1) + 24\Gamma(c+r+5)}{\theta^r (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))} \quad (10)$$

By substituting $r = 1$ in equation (10), we will get the mean of weighted XRani distribution

$$E(X) = \mu_1' = \frac{\theta^{10} \Gamma(c+2) + 48\theta^5 \Gamma(c+2) + 24\Gamma(c+6)}{\theta(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))}$$

Now by substituting $r = 2, 3$ and 4 in equation (10), we will get the other three moments of weighted XRani distribution as

$$E(X^2) = \mu_2' = \frac{\theta^{10} \Gamma(c+3) + 48\theta^5 \Gamma(c+3) + 24\Gamma(c+7)}{\theta^2 (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))}$$

$$E(X^3) = \mu_3' = \frac{\theta^{10} \Gamma(c+4) + 48\theta^5 \Gamma(c+4) + 24\Gamma(c+8)}{\theta^3 (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))}$$

$$E(X^4) = \mu_4' = \frac{\theta^{10} \Gamma(c+5) + 48\theta^5 \Gamma(c+5) + 24\Gamma(c+9)}{\theta^4 (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))}$$

$$\text{Variance} = \frac{\theta^{10} \Gamma(c+3) + 48\theta^5 \Gamma(c+3) + 24\Gamma(c+7)}{\theta^2 (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))} - \left(\frac{\theta^{10} \Gamma(c+2) + 48\theta^5 \Gamma(c+2) + 24\Gamma(c+6)}{\theta (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))} \right)^2$$

$$S.D(\sigma) = \sqrt{\left(\frac{\theta^{10} \Gamma(c+3) + 48\theta^5 \Gamma(c+3) + 24\Gamma(c+7)}{\theta^2 (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))} - \left(\frac{\theta^{10} \Gamma(c+2) + 48\theta^5 \Gamma(c+2) + 24\Gamma(c+6)}{\theta (\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5))} \right)^2 \right)}$$

5.2 Harmonic mean

The harmonic mean of performed distribution can be determined as

$$H.M = E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} f_w(x; \theta, c) dx$$

$$H.M = \int_0^{\infty} \frac{1}{x} \frac{x^c \theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$H.M = \int_0^{\infty} \frac{x^{c-1} \theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$H.M = \frac{\theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \int_0^{\infty} x^{c-1} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$H.M = \frac{\theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \left(\begin{aligned} &\theta^6 \int_0^{\infty} x^{(c+1)-2} e^{-\theta x} dx + 48\theta \int_0^{\infty} x^{(c+1)-2} e^{-\theta x} dx \\ &+ 24 \int_0^{\infty} x^{(c+4)-1} e^{-\theta x} dx \end{aligned} \right) \quad (11)$$

After the simplification of equation (11), we get

$$H.M = \frac{\theta \left(\theta^9\Gamma(c+1) + 48\theta^4\Gamma(c+1) + 24\Gamma(c+4) \right)}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5) \right)}$$

5.3 Moment generating function and characteristic function

Consider X be the random variable which follows weighted XRani distribution, then the moment generating function of explored distribution can be determined as

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f_w(x; \theta, c) dx$$

$$M_X(t) = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(tx)^k}{k!} f_w(x; \theta, c) dx$$

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} x^k f_w(x; \theta, c) dx$$

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu_k$$

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{\theta^{10}\Gamma(c+k+1) + 48\theta^5\Gamma(c+k+1) + 24\Gamma(c+k+5)}{\theta^k \left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5) \right)} \right)$$

$$M_X(t) = \frac{1}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5) \right)} \sum_{k=0}^{\infty} \frac{t^k}{k! \theta^k} \left(\theta^{10}\Gamma(c+k+1) + 48\theta^5\Gamma(c+k+1) + 24\Gamma(c+k+5) \right)$$

Obviously the characteristic function of weighted XRani distribution can be obtained as

$$\varphi_X(t) = M_X(it)$$

$$M_X(it) = \frac{1}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \sum_{k=0}^{\infty} \frac{it^k}{k!} \left(\theta^{10}\Gamma(c+k+1) + 48\theta^5\Gamma(c+k+1) + 24\Gamma(c+k+5)\right)$$

6. Bonferroni and Lorenz Curves

The bonferroni and Lorenz curves were employed in economics to determine the graphical representation of the distribution of income or wealth. The proposed curves are old classical curves and nowadays being applied in different fields. The presented curves are defined as

$$B(p) = \frac{1}{p\mu_1'} \int_0^q x f_w(x; \theta, c) dx$$

$$L(p) = pB(p) = \frac{1}{\mu_1'} \int_0^q x f_w(x; \theta, c) dx$$

$$\text{Here } \mu_1' = \frac{\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)}{\theta(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))}$$

$$B(p) = \frac{\theta(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))}{p\left(\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)\right)} \int_0^q \frac{x^c \theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))}{p\left(\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)\right)} \int_0^q \frac{x^{c+1} \theta^{c+5}}{\left(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)\right)} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta^{c+6}}{p\left(\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)\right)} \int_0^q x^{c+1} \left(\theta^6 + 48\theta + 24x^4\right) e^{-\theta x} dx$$

$$B(p) = \frac{\theta^{c+6}}{p\left(\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)\right)} \left(\theta^6 \int_0^q x^{c+1} e^{-\theta x} dx + 48\theta \int_0^q x^{c+1} e^{-\theta x} dx + 24 \int_0^q x^{c+5} e^{-\theta x} dx \right) \quad (12)$$

$$\text{By Putting } \theta x = t \Rightarrow \theta dx = dt \Rightarrow dx = \frac{dt}{\theta}$$

$$\text{Now } x = \frac{t}{\theta}$$

$$\text{As } x \rightarrow 0, t \rightarrow 0 \text{ and as } x \rightarrow q, t \rightarrow \theta q$$

After the simplification of equation (12), we obtain

$$B(p) = \frac{1}{p\left(\theta^{10}\Gamma(c+2) + 48\theta^5\Gamma(c+2) + 24\Gamma(c+6)\right)} \left(\theta^{10}\gamma(c+2, \theta q) + 48\theta^5\gamma(c+2, \theta q) + 24\gamma(c+6, \theta q) \right)$$

$$L(p) = \frac{1}{\left(\theta^{10} \Gamma(c+2) + 48\theta^5 \Gamma(c+2) + 24\Gamma(c+6) \right)} \left(\theta^{10} \gamma(c+2, \theta q) + 48\theta^5 \gamma(c+2, \theta q) + 24\gamma(c+6, \theta q) \right)$$

7. Maximum Likelihood Estimation and Fisher's Information Matrix

In this portion, we have derived the method of maximum likelihood estimation to estimate the parameters of weighted XRani distribution. Consider the random sample $X_1, X_2, \dots, X_n$ of size n from the proposed weighted XRani distribution, then the likelihood function given by

$$L(x) = \prod_{i=1}^n f_w(x; \theta, c)$$

$$L(x) = \prod_{i=1}^n \left(\frac{x_i^c \theta^{c+5}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \left(\theta^6 + 48\theta + 24x_i^4 \right) e^{-\theta x_i} \right)$$

$$L(x) = \frac{\theta^{n(c+5)}}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)^n} \prod_{i=1}^n \left(x_i^c \left(\theta^6 + 48\theta + 24x_i^4 \right) e^{-\theta x_i} \right)$$

The log likelihood function can be stated as

$$\begin{aligned} \log L &= n(c+5) \log \theta - n \log \left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right) \\ &+ c \sum_{i=1}^n \log x_i + \sum_{i=1}^n \log \left(\theta^6 + 48\theta + 24x_i^4 \right) - \theta \sum_{i=1}^n x_i \end{aligned} \quad (13)$$

The maximum likelihood estimate of θ and c can be obtained by differentiating the log likelihood equation (13) with respect to parameter θ and c . The following normal equations must be satisfied

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+5)}{\theta} - n \left(\frac{10\theta^9 \Gamma(c+1) + 240\theta^4 \Gamma(c+1)}{\left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right)} \right) + \sum_{i=1}^n \left(\frac{(6\theta^5 + 48)}{\left(\theta^6 + 48\theta + 24x_i^4 \right)} \right) - \sum_{i=1}^n x_i = 0$$

$$\frac{\partial \log L}{\partial c} = n \log \theta - n \psi \left(\theta^{10} \Gamma(c+1) + 48\theta^5 \Gamma(c+1) + 24\Gamma(c+5) \right) + \sum_{i=1}^n \log x_i = 0$$

The above system of non-linear equations is too difficult to solve it algebraically. Therefore we use the technique like Newton-Raphson method to estimate the required parameters of developed distribution.

In order to apply the asymptotic normality results for obtaining the confidence interval. We state that $\hat{\alpha} = (\hat{\theta}, \hat{c})$ which represents the MLE of $\alpha = (\theta, c)$. The results can be interpreted as

$$\sqrt{n}(\hat{\alpha} - \alpha) \rightarrow N_2(0, I^{-1}(\alpha))$$

Here $I^{-1}(\alpha)$ is Fisher's information matrix i.e.,

$$I(\alpha) = -\frac{1}{n} \begin{pmatrix} E \left(\frac{\partial^2 \log L}{\partial \theta^2} \right) & E \left(\frac{\partial^2 \log L}{\partial \theta \partial c} \right) \\ E \left(\frac{\partial^2 \log L}{\partial c \partial \theta} \right) & E \left(\frac{\partial^2 \log L}{\partial c^2} \right) \end{pmatrix}$$

Here, we can state that

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{n(c+5)}{\theta^2} - n \left(\frac{(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5)) (90\theta^8\Gamma(c+1) + 960\theta^3\Gamma(c+1))}{(10\theta^9\Gamma(c+1) + 240\theta^4\Gamma(c+1))^2} - \frac{(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))^2}{(10\theta^9\Gamma(c+1) + 240\theta^4\Gamma(c+1))^2} \right)$$

$$+ \sum_{i=1}^n \left(\frac{(\theta^6 + 48\theta + 24x_i^4) (30\theta^4) - (6\theta^5 + 48)^2}{(\theta^6 + 48\theta + 24x_i^4)^2} \right)$$

$$E\left(\frac{\partial^2 \log L}{\partial c^2}\right) = -n\psi'(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial c}\right) = \frac{n}{\theta} - n\psi' \left(\frac{(10\theta^9\Gamma(c+1) + 240\theta^4\Gamma(c+1))}{(\theta^{10}\Gamma(c+1) + 48\theta^5\Gamma(c+1) + 24\Gamma(c+5))} \right)$$

Hence α is not known so we estimate $I^{-1}(\alpha)$ by $I^{-1}(\hat{\alpha})$ and this can be applied to obtain the asymptotic confidence interval of θ and c .

8. Application

In this portion, we have performed and assessed the goodness of fit of weighted XRani distribution by analyzing and investigating a real lifetime data set and then comparison has been developed in order to reveal that the weighted XRani distribution provides a better fit over XRani, Rani, Shanker, Akash and Lindley distributions. The following real lifetime data set is. The given below real symmetric data set consists of the failure times of windshields and the considered real lifetime data set was discussed by Murthy *et al.* (2004) [8]. The observations of real data set is

Table 1: Data represents the failure times of windshields by Murthy *et al.* (2004) [8]

0.04	0.3	0.31	0.557	0.943	1.07	1.124	1.248	1.281	1.281
1.303	1.432	1.48	1.51	1.51	1.568	1.615	1.619	1.652	1.652
1.757	1.795	1.866	1.876	1.899	1.911	1.912	1.9141	0.981	2.010
2.038	2.085	2.089	2.097	2.135	2.154	2.190	2.194	2.223	2.224
2.23	2.3	2.324	2.349	2.385	2.481	2.610	2.625	2.632	2.646
2.661	2.688	2.823	2.89	2.9	2.934	2.962	2.964	3	3.1
3.114	3.117	3.166	3.344	3.376	3.385	3.443	3.467	3.478	3.578
3.595	3.699	3.779	3.924	4.035	4.121	4.167	4.240	4.255	4.278
4.305	4.376	4.449	4.485	4.570	4.602	4.663	4.694		

In order to evaluate the model criterion values and estimate the unknown parameters, the analysis is performed and carried out through the R software. To inquire the performance of weighted XRani distribution over XRani, Rani, Shanker, Akash and Lindley distribution, the suggested criterion values such as *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion), *AICC* (Akaike Information Criterion Corrected) and $-2\log L$ are applied. The distribution is good even if it has the smaller criterions in comparison over other compared distributions. The observed criterion values are also determined through the following formulas

$$AIC = 2k - 2\log L, \quad BIC = k \log n - 2\log L \quad \text{and} \quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$$

Where n is the sample size, k is the number of parameters in statistical model and $-2\log L$ is maximized value of log-likelihood function. It is quite evident and distinct from result given above in table 2 that the weighted XRani distribution has the smaller *AIC*, *BIC*, *AICC* and $-2\log L$ criterion values in comparison over XRani, Rani, Shanker, Akash and Lindley distributions which implies that the weighted XRani distribution provides a significant fit over XRani, Rani, Shanker, Akash and Lindley distributions.

Table 2: Shows Interpretation and Effectiveness of Fitted Distributions

Distributions	MLE	S.E	-2logL	AIC	BIC	AICC
Weighted XRani	$\hat{\theta} = 2.2251456$	$\hat{\theta} = 0.2058926$	284.5307	288.5307	293.4854	288.6718
	$\hat{c} = 2.2148864$	$\hat{c} = 0.6057687$				
XRani	$\hat{\theta} = 1.38138982$	$\hat{\theta} = 0.04632054$	315.4194	317.4194	319.8968	317.4659
Rani	$\hat{\theta} = 1.51450004$	$\hat{\theta} = 0.05536443$	298.1371	300.1371	302.6144	300.1836
Shanker	$\hat{\theta} = 0.64302655$	$\hat{\theta} = 0.04505049$	314.4891	316.4891	318.9664	316.5356
Akash	$\hat{\theta} = 0.93208005$	$\hat{\theta} = 0.05520992$	306.3466	308.3466	310.824	308.3931
Lindley	$\hat{\theta} = 0.62835578$	$\hat{\theta} = 0.04923216$	319.454	321.454	323.9313	321.5005

9. Conclusion

In the proposed manuscript, we have decorated a new extension of XRani distribution called as weighted XRani distribution has been instigated and evolved. The explored distribution has been framed by implementing the weighted technique to XRani distribution and the performed new distribution has been elucidated with different statistical characteristics via shape of behavior of pdf and cdf, moments, mean and variance, reliability function, hazard rate function, reverse hazard rate function, bonferroni and lorenz curves, moment generating function and order statistics has been described. In addition, the parameters of executed distribution are estimated through method of maximum likelihood estimation. Furthermore, a real lifetime data set has been considered and fitted to examine the predictability and superiority of developed new distribution and hence it is clearly perceived from result that the weighted XRani distribution leads to a better fit in comparison to XRani, Rani, Shanker, Akash and Lindley distributions.

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