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A study on symmetry of second derivatives with symmetric functions

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Abstract

The link between the two results is their equivalence, which is established in Theorem. For the sake of simplicity we confine our discussion to functions defined on open sets of the plane and to second order mixed partial derivatives. The interested reader can easily extend the obtained result to more general situations. All functions of two variables are assumed to be continuous ("jointly continuous" is the terminology preferred by some authors). Marsden and Hoffman use the Mean Value Theorem to give a plausible proof of the equality $xy = yx$ while emphasizing its nonintuitive nature. Kaplan derives the same equality from Fubini's Theorem in the case when f , f_x , f_y and f_{xy} are continuous. A first order differential equation: $M(x,y)+N(x,y)(dy/dx)=0$, often written as $M(x,y)dx + N(x,y)dy = 0$, is said to be exact, over a domain D in $\mathbb{R}^2$, if there exists a continuously differentiable function $f(x,y)$, such that the functions M and N are the partial derivative functions of $f(x,y)$, with respect to x and y respectively, i.e. $M(x,y)=(f_x)(x,y)$ and $N(x,y)=(f_y)(x,y)$ for every (x,y) in D .

The most important characterization of exact equations requires that the above function $f(x,y)$ have identically equal mixed second order derivatives $f_{xy}=f_{yx}$. A standard sufficient condition for the above is continuity of second order partial derivatives. This condition states that the above condition is exact if and only if the first order partials $M_y(x,y)=N_x(x,y)$ for every (x,y) in D .

Keywords: Theorem, derivatives, variable, equations

Introduction

Suppose f is a real-valued function of two variables x,y and $f(x,y)$ is defined on an open subset of $U \subset \mathbb{R}^2$. Suppose further that both the second-order mixed partial derivatives $f_{xy}(x,y)$ and $f_{yx}(x,y)$ exist and are continuous on U . Then, we have:

$$f_{xy} = f_{yx}$$

on all of U .

General statement

The statement can be generalized in two ways:

- We can generalize it to higher-order partial derivatives.
- We can generalize it to functions of more than two variables.

Suppose f is a function of two variables which we denote x and y . There are two possible second-order mixed partial derivative functions for f , namely f_{xy} and f_{yx} . In most ordinary situations, these are equal by Clairaut's theorem on equality of mixed partials. Technically, however, they are defined somewhat differently.

Often the term mixed partial is used as shorthand for the second-order mixed partial derivative. However, mixed partial may also refer more generally to a higher partial derivative that involves differentiation with respect to multiple variables.

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Baire classification of symmetric derivatives

Let f be an extended real function, that is, a mapping from $\mathbb{R}$ into the extended real line. If there exists a sequence of continuous real functions, $f_1, f_2, \dots$ such that $f_n(x) \rightarrow f(x)$ for each $x \in \mathbb{R}$, then f is said to be in the first Baire class, which is denoted by B_1 . Theorem 2.1. If $f \in \Sigma$, then $f^s \in B_1$.

Proof. Let $f \in \Sigma$ and P be a perfect subset of $\mathbb{R}$. Choose real numbers α, β, γ and δ such that $\alpha < \beta < \gamma < \delta$. According to Preiss [14, Theorem 1], it suffices to show that the sets $A = \{x \in P: f^s(x) \leq \alpha\}$ and $D = \{x \in P: f^s(x) \geq \delta\}$ cannot both be dense in P . To do this, assume both A and D are dense in P in order to arrive at a contradiction.

Define $B = \{x \in P: f^s(x) > \beta\}$ and $C = \{x \in P: f^s(x) < \gamma\}$.

Since $P = B \cup C$, at least one of the sets B or C must be a second category subset of P . Suppose first that B is a second category subset of P . Through the addition of an appropriate linear function to f , it may be assumed, without loss of generality, that $\beta = 0$. Then define

$$B_n = \{x \in P: f(x+h) > f(x-h), h \in (0, 1/n)\}.$$

Since $B \subseteq \bigcup_{n=1}^{\infty} B_n$ and B is a second category subset of P , there exists a closed interval I and an integer n_0 such that $I \cap P$ is perfect and B_{n_0} is dense in $I \cap P$.

Suppose $x \in I \cap P \cap A$. Then, since $\alpha < 0$, $x \in A$ implies that there is a $\delta(x) \in (0, 1/n_0)$ such that $f(x+h) - f(x-h) < h\alpha$ whenever $0 < h < \delta(x)$. Since $I \cap P$ is perfect, x is either a right or a left limit point of $I \cap P$. We assume, without loss of generality, that x is a right limit point. Since A is dense in $I \cap P$, there exists a $y \in A \cap (x, x + \delta(x))$ and a $\delta(y) > 0$ such that $f(y+h) - f(y-h) < h\alpha < 0$ when $0 < h < \delta(y)$. B_{n_0} is also dense in $I \cap P$, so a $z \in (x, y) \cap (x, x + \delta(y)) \cap B_{n_0}$ may be chosen. Then, using the inequalities $0 < y - z < \delta(x)$, $0 < z - x < \delta(y)$ and $0 < y - x < 1/n_0$ we obtain the relations

$$\begin{aligned} 0 &> \alpha(y-z) > f(x+(y-z)) - f(x-(y-z)) \\ &= -[f(y+(z-x)) - f(y-(z-x))] + [f(z+(y-x)) - f(z-(y-x))] \\ &> -\alpha(z-x) > 0. \end{aligned}$$

This obvious contradiction leads us to conclude that B cannot be a second category subset of P . A similar contradiction is reached if we assume that C is a second category subset of P . Thus, we are forced to conclude that A and D cannot both be dense in P . $\square$

A more general derivative than the symmetric derivative could be defined as follows: Let α be a positive function defined on a neighbourhood of 0. Set

$$f_{\alpha}^s(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{\alpha(h)},$$

and let Σ_{α} be the class of all real functions for which exists everywhere. Then an argument similar to the proof of Theorem 2.1 shows that if $f \in \Sigma_{\alpha}$, then $f_{\alpha}^s \in \mathcal{B}_1$. Although it is unknown whether each $f \in \Sigma$ is measurable, the next theorem does show something about the behavior of such a function and at the same time shows that not every $\mathcal{B}_1$ -function is a symmetric derivative

Characterizations of $m\Sigma$: To a real function f we associate the sets $C(f) = \{x: f \text{ is continuous at } x\}$ and $D(f) = \{x: f(x) \text{ exists and is finite}\}$. The following theorem gives characterizations of a function $f \in m\Sigma$ in terms of these sets.

Theorem 3.1. $f \in m\Sigma$, then the following are equivalent:

- $f \in m\Sigma$;
- $D(f)$ has full measure;
- $C(f)$ is dense.

This theorem is an immediate consequence of the following lemma.

Lemma Let f be a real function such that a.e. either $f^s(x) \neq -\infty$ or $f^s(x) \neq \infty$. Then the following statements are equivalent

- f is measurable;
- $D(f)$ has full measure;
- $C(f)$ is dense.

Proof. That (a) implies (b) is Khintchine's theorem [9, p. 217]. That (b) implies both (c) and (a) is obvious. To show that (c) implies (b) we assume $C(f)$ is dense.

According to Belna, Evans and Humke [2, Theorem 2], there exists a set E of full measure (in fact E^c is σ -porous) such that for each $x \in E$ we have

$$f^s(x) = \min\{D_+f(x), D_-f(x)\} \quad \text{and} \quad \bar{f}^s(x) = \max\{D^+f(x), D^-f(x)\},$$

where Df, D^-f, D_+f, D^+f denote the Dim derivatives of f . If E^* is the set of points in E where either $f^s \neq -\infty$ or $f^s \neq \infty$, then it follows from the Denjoy-Young-Saks theorem (see [3, p. 65]) that f' exists and is finite a.e. in E^* . Thus (c) implies (b).

The structure of functions in $m\Sigma$: In this section we use the concepts of reflection and local symmetry. Suppose $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. The reflection of A through x is given by $\mathcal{R}_x(A) = \{2x - a : a \in A\}$. Then A is said to be locally symmetric at x if for some $\delta > 0$

$$\mathcal{R}_x(A \cap (x - \delta, x + \delta)) = A \cap (x - \delta, x + \delta),$$

or equivalently, if the characteristic function of A is symmetrically continuous at x . We refer to A as a locally symmetric set if it is locally symmetric at each $x \in \mathbb{R}$.

Lemma 4.1. $f \in m\Sigma$, then the sets

$$A = \left\{x : \limsup_{t \rightarrow x} f(t) = \infty\right\} \quad \text{and} \quad B = \left\{x : \liminf_{t \rightarrow x} f(t) = -\infty\right\}$$

are both countable and closed.

Proof. We prove the lemma in the case of A . The assertion for B then follows by considering $-f$.

It is clear that A is closed. Because $C(f)$ is dense (Theorem 3.1), A must be nowhere dense. Since A is closed we may write $A = P \cup N$, where P is perfect and N is countable. Suppose $P \neq \emptyset$, and let (α, β) be a component of P^c (α or β could be infinite). Since $P \neq \emptyset$, α or β must be finite. Suppose β is finite. Then, since P is closed, $\beta \in P$. P being perfect and $(\alpha, \beta) \subset P^c$, we see that for each $\delta > 0$, $[\beta, \beta + \delta) \cap P$ is uncountable.

Since $A \cap (\alpha, \beta) \subset N$ is countable, we may choose a sequence, $\{\beta_n\} \subset P$, such that β_n decreases to β and $\mathcal{R}_\alpha(\{\beta_n\}) \cap A = \emptyset$. Because $\beta_n \in A$ and $\mathcal{R}_\beta(\beta_n) \notin A$, we may choose a $t_n > \beta$ such that $|t_n - \beta_n| < 1/n$ and $f(t_n) - f(\mathcal{R}_\beta(t_n)) > n$. Consequently $f(t_n) \rightarrow \beta$ and $f^s(\beta) = \infty$. Similarly, if $\alpha > -\infty$, then $f^s(\alpha) = -\infty$.

Now we note that $P^c = \bigcup_{n=1}^{\infty} (\alpha_n, \beta_n)$ where each (α_n, β_n) is a component of P^c . Since P is a nowhere dense set in $\mathbb{R}$, the sets $\{\alpha_n\}$ and $\{\beta_n\}$ are both dense in P . This implies, from the above, that the sets

$$I^+ = \{x : f^s(x) = \infty\} \quad \text{and} \quad I^- = \{x : f^s(x) = -\infty\}$$

are disjoint and dense in P ; furthermore, according to Theorem 2.1, I^+ and I^- are G_δ sets. This contradicts the fact that P is a Baire space.

For any $f \in m\Sigma$, we let

$$M_f = \left\{x : \left| \limsup_{\substack{t \rightarrow x \\ t \in C(f)}} f(t) \right| = \infty\right\}$$

We define μ_f to be the real function

$$\mu_f(x) = \begin{cases} \limsup_{\substack{t \rightarrow x \\ t \in C(f)}} f(t), & x \in M_f^c, \\ f(x), & x \in M_f. \end{cases}$$

We also define the extended symmetric derivative of μ_f at x to be

$$\mu_f^*(x) = \lim_{(h \rightarrow 0)} \frac{\mu_f(x+h) - \mu_f(x-h)}{2h},$$

where $(h \rightarrow 0)$ means that $h \rightarrow 0$ through the set $\{h : x \pm h \notin M_f\}$. (Note that the extended symmetric derivative equals the symmetric derivative whenever the latter exists)

Conclusion

Jansen gave explicit formulas for the computation of the second-order derivatives of the elementary symmetric functions. But they are only applicable to those pairs of items which have unequal parameters.

It is shown here that for equal parameters similar explicit formulas do exist, too, facilitating the application of the Newton-Raphson procedure to estimate the parameters in the Rasch model and related models according to the conditional maximum likelihood principle. A similar calculation for the other mixed partial yields that:

$$f_{yx}(x_0, y_0) = \lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} \left[\frac{f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0)}{(y - y_0)(x - x_0)} \right]$$

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