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The G-Bi-shadowing property with nilpotent group

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Abstract

In this paper, we investigated the property of bi-shadowing for actions of certain finitely generated groups. We have given a definition of the term Bi-Conditions and mentioning some key definitions with example. We have formulated the text of the reductive bi-shadowing theory (through which we set sufficient conditions for the action to have the bi-shadowing property) and we try to prove it by formulating and proving some lemmas that complement one another in order to arrive at proving the main theorem in the paper.

Keywords: Group action, shadowing, inverse shadowing, bi-shadowing, virtually nilpotent group

Introduction

The shadowing property is considered one of the important properties in the study of dynamic systems because it has multiple applications of importance in different fields of life. This is what made scientists continue to study it and develop it into the inverse shadowing property and the bi-shadowing property, as well as studying it in different spaces. This paper aims to study the G-Bi shading property (which means the binary shading property in group space) in the Nilpotent group.

In the second chapter of this paper, we present the basic definitions, including the shadowing property, the inverse shadowing property, and the bi-shadowing property in group space. You can review ^[1, 2, 3, 4] for more details.

In the third chapter of this paper, we aim to formulate the text of the reductive bi-shadowing theory (through which we set sufficient conditions for the action to have the bi-shadowing property) and we try to prove it by formulating and proving some lemmas that complement each other to arrive at proof of the main theory in the paper. This result will be motivated by a similar result for shadowing and inverse shadowing that was proven in ^[5, 21].

Preliminaries

In this section, we will discuss a set of definitions that we will need in the body of the paper, some of them are basic definitions that you need to form the basic structure of theories, and others we will need to prove those theories.

Considering that the definitions of the group and subgroup are trivial, below we will present the definition of the normal subgroup, commutator, Abelian group, nilpotent group and the virtually nilpotent.

Definitions 2.1 ^[2, 51]: Assume that $\mathbb{C}$ is a group and $\mathbb{E}$ is a normal subgroup in $\mathbb{C}$ (write $\mathbb{E} \triangleleft \mathbb{C}$) which satisfy $c\mathbb{E} = \mathbb{E}c$, for $c \in \mathbb{C}$.

And for $c' \in \mathbb{C}$ then a commutator of c and c' satisfy: $[c, c'] = cc'c^{-1}c'^{-1}$

If $\mathbb{E}_1, \mathbb{E}_2$ are subgroups of $\mathbb{C}$, then a commutator of them $[\mathbb{E}_1, \mathbb{E}_2]$ is generated by $[c_1, c_2]$, where $c_1 \in \mathbb{E}_1, c_2 \in \mathbb{E}_2$.

The group $\mathbb{C}$ is called nilpotent (class n) if there exist n of subgroups $\mathbb{E}_1, \dots, \mathbb{E}_{n+1}$ satisfy

$$\mathbb{C} = \mathbb{E}_1 \triangleright \dots \triangleright \mathbb{E}_{n+1} = e, \text{ for } \mathbb{E}_n \neq e, \text{ and } \mathbb{E}_{i+1} = [\mathbb{E}_i, \mathbb{C}].$$

Therefore; the Abelian group is nilpotent (class 1). Let a group $\mathbb{C}$ is a finitely generated then we called virtually nilpotent if we find subgroup $\mathbb{E}$ of $\mathbb{C}$ which is a normal nilpotent and have finite index (i.e. $\mathbb{C}/\mathbb{E}$ is finite (factor group)).

Example 2.2 ^[6, 7]: Heisenberg group is nilpotent group, and Polynomial growth group is virtually nilpotent.

In the following definitions, theories and proofs, we will suppose that the group action $\Psi: \mathbb{C} \times \mathbb{S} \rightarrow \mathbb{S}$ of a finitely generated group $\mathbb{C}$ on a metric group space $(\mathbb{S}, \mathbb{d})$ and the properties with respect to the generating set $\mathbb{W}$, (any generating set of $\mathbb{C}$) for short we write (W. R. T.).

Definition 2.3 ^[2]: The continuity of an action Ψ is called a uniform if for some generating set $\mathbb{W}$, any $\Psi(w, \cdot), w \in \mathbb{W}$, is uniformly continuous. And in this paper we suppose the continuity of an action Ψ is uniform.

Below we mention the shadowing definition, therefore, we need to provide the definition of the pseudo-orbit. For $\rho > 0$, the concept ρ -pseudo-orbit of an action Ψ refers to the sequence $\{y_c: c \in \mathbb{C}\}$ that satisfies: $\mathbb{d}(\gamma_{wc}, \Psi_w(y_c)) < \rho, w \in \mathbb{W}, c \in \mathbb{C}$. ^[5]

Definition 2.4 ^[5]: The concept shadowing property on a set $\mathbb{M} \subset \mathbb{S}$ w.r.t. $\mathbb{W}$ for short we write (S.P.M.W.) refers to an action Ψ that satisfies: if for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any ρ -pseudo-orbit $\{x_c \in \mathbb{M}\}$ w.r.t. $\mathbb{W}$ there exists an exact-orbit $\{p_c = \Psi(c, p_e): c \in \mathbb{C}\}$ of Ψ satisfying the inequalities

$$\mathbb{d}(p_c, x_c) < \varepsilon, c \in \mathbb{C}. (1)$$

Proposition 2.5 ^[5]: Assume that $\mathbb{W}$ and $\mathbb{W}'$ are finite, symmetric, generating sets of $\mathbb{C}$. An action Ψ has S.P.M.W., if and only if it has S.P.M.W'.

Below we mention the inverse shadowing definition, therefore, we need to provide the definition of the method and the orbit of it. For $\rho > 0$, the concept ρ -method for Ψ w.r.t. $\mathbb{W}$ refers to a family $\Gamma = \{\gamma_c \in \mathcal{C}(\mathbb{S}, \mathbb{S}): c \in \mathbb{C}\}$ that satisfies:

$$\mathbb{d}(\gamma_{wc}(x), \Psi(w, \gamma_c(x))) < \rho, c \in \mathbb{C}, w \in \mathbb{W}, x \in \mathbb{S}; (2)$$

And an orbit of the method Γ is a family $\{\chi_c = \gamma_c(x_e) \in \mathbb{S}: c \in \mathbb{C}\}$. (for $\gamma_e = \text{Id}$)

Definition 2.6 ^[2]: The concept inverse shadowing property w.r.t. $\mathbb{W}$ for short we write (I.S.P.W.) refers to an action Ψ that satisfies: if for any $\varepsilon > 0$ there exists a $\rho > 0$ such that for any $p \in \mathbb{S}$ and any ρ -method Γ with respect to the generating set $\mathbb{W}$ there exists an orbit $\{\chi_c\}$ of Γ satisfying

$$\mathbb{d}(\Psi(c, p), \chi_c) < \varepsilon, c \in \mathbb{C}.$$

Below we mention the bi-shadowing definition, therefore, we need to provide the definition of distance between two actions. For Ψ and φ are actions, the distance between them is given by:

$$\mathbb{d}_0(\Psi, \varphi) = \sup_{x \in \mathbb{X}} \{\mathbb{d}(\Psi(s, x), \varphi(s, x))\} \text{ for } s \in \mathbb{S}.$$

Definition 2.7 ^[3]: The concept bi-shadowing property w.r.t. $\mathbb{W}$ for short we write (B.S.P.W.) refers to an action Ψ that satisfies: if for any $\varepsilon > 0$ there exists $\alpha > 0$ and $0 < \delta \leq \varepsilon/\alpha$ such that for any δ -pseudo-orbit $\{y_c: c \in \mathbb{C}\}$ of Ψ and any action $\varphi \in \mathcal{C}(\mathbb{S}, \mathbb{S})$ satisfying $\mathbb{d}_0(\Psi, \varphi) \leq \varepsilon/\alpha - \delta$ then there exists an orbit $\{x_c: c \in \mathbb{C}\}$ of φ such that

$$\mathbb{d}(x_c, y_c) \leq \alpha(\delta + \mathbb{d}_0(\Psi, \varphi)) \leq \varepsilon, \text{ for all } c \in \mathbb{C},$$

Below we mention the expansive definition

Definition 2.8 ^[5]: The concept expansive property on a set $\mathbb{M} \subset \mathbb{S}$ for short we write (E.P.M.) refers to an action Ψ that satisfies: if there exists $a > 0$ such that if $\Psi(c, x), \Psi(c, y) \in \mathbb{M}$, $\mathbb{d}(\Psi(c, x), \Psi(c, y)) \leq a, x, y \in \mathbb{S}, c \in \mathbb{C}$, then $x = y$.

We now present the main definition in this paper (Bi-Conditions), which brings together the main conditions for achieving it.

Definition 2.9: Consider two sets $\mathbb{M}, \mathbb{N} \subset \mathbb{S}$. We say that an action Ψ satisfies the Bi-Conditions w.r.t. $(\mathbb{M}, \mathbb{N})$ for short we write (B.C.M, N) if the following conditions are satisfied:

- C1- there exists $\gamma > 0$ such that $B(\gamma, \mathbb{N}) \subset \mathbb{M}$;
- C2- Ψ has (S.P.N.);
- C3- Ψ has (I.S.P.);
- C4- Ψ is (E.P.M.).

Main Theorem

The following establish the reductive bi-shadowing theorem (RBST), which will provide sufficient conditions for an action of a virtually nilpotent group to possess the bi-shadowing property.

Theorem. Let Ψ be an action of a finitely generated virtually nilpotent group $\mathbb{C}$. Assume that there exists $c \in \mathbb{C}$ such that $\Psi|_{\langle c \rangle}$ satisfies (B.C.M, N). Then an action Ψ has (B.S.P.).

To prove the above theorem, we will present some interconnected lemmas and prove them to arrive at the final proof. Some results, for brevity, we will refer to in other sources and others we will prove.

Lemma 1. Let $\mathbb{E}$ be finitely generated normal subgroup of $\mathbb{C}$. If $\Psi|_{\mathbb{E}}$ satisfies (B.C.M, $\mathbb{N}$). Then an action Ψ satisfies (B.C.M, $\mathbb{N}$).

Proof:

Chose a finitely symmetric generate set $\mathbb{W}_{\mathbb{E}}$ in $\mathbb{E}$ and also it to a finitely symmetric generate set $\mathbb{W}$ in $\mathbb{C}$. By using Proposition 2.5, suppose that the generating set $\mathbb{W}$ was chose in the same way.

To prove an action Ψ satisfies (B.C.M, $\mathbb{N}$) we must prove an action Ψ satisfies C1, C2, C3, and C4 in Definition 2.9.

C1 is trivial. To prove C2, we must prove an action Ψ has shadowing property, let $\Delta, \gamma > 0$ be the constants from the definitions of (B.C.M, $\mathbb{N}$) for $\Psi|_{\mathbb{E}}$. Since the continuity of an actions $\{\Psi(\mathbb{w}, \cdot) : \mathbb{w} \in \mathbb{W}\}$ are uniform, there exists $\delta < \min(\Delta/3, \gamma)$ such that

$$\mathbb{d}(\Psi(\mathbb{w}, x), \Psi(\mathbb{w}, y)) < \Delta/3 \quad (3)$$

for any $\mathbb{w} \in \mathbb{W}$ and any two points $x, y \in \mathbb{S}$ satisfying $\mathbb{d}(x, y) < \delta$. Fix $\varepsilon \in (0, \delta)$ and choose $\rho < \varepsilon$ from the definition of shadowing for $\Psi|_{\mathbb{E}}$ for the generating set $\mathbb{W}_{\mathbb{E}}$. Fix a ρ -pseudo-orbit $\{y_{\mathbb{c}} \in \mathbb{N} : \mathbb{c} \in \mathbb{C}\}$ of Ψ .

For each element $q \in \mathbb{C}$ if we take the sequence $\{z_{\mathbb{e}'} = y_{\mathbb{e}'q} : \mathbb{e}' \in \mathbb{E}\}$. We note that this sequence is a ρ -pseudo-orbit of $\Psi|_{\mathbb{E}}$. And since $\Psi|_{\mathbb{E}}$ satisfies (B.C.M, $\mathbb{N}$), then there exists a unique point $x_q \in \mathbb{M}$ such that

$$\mathbb{d}(z_{\mathbb{e}'}, \Psi(\mathbb{e}', x_q)) = \mathbb{d}(y_{\mathbb{e}'q}, \Psi(\mathbb{e}', x_q)) < \varepsilon, \mathbb{e}' \in \mathbb{E}. \quad (4)$$

This existence of x_q can be from (C2), and the uniqueness from (C1), (C4), and the inequality $\varepsilon < \gamma$.

Let us prove the sequence $\{x_q : q \in \mathbb{C}\}$ is an exact-orbit.

Fix $\mathbb{w} \in \mathbb{W}$ and $q \in \mathbb{C}$. Consider an arbitrary element $\mathbb{e} \in \mathbb{E}$. Since $\mathbb{E}$ is a normal subgroup of $\mathbb{C}$, there exists an element $\mathbb{e}' \in \mathbb{E}$ such that

$$\mathbb{w}\mathbb{e}' = \mathbb{e}\mathbb{w}. \quad (5)$$

It follows from (3)–(5) that

$$\mathbb{d}(y_{\mathbb{w}\mathbb{e}'q}, \Psi(\mathbb{e}, x_{\mathbb{w}q})) < \varepsilon \quad (6)$$

$$\mathbb{d}(\Psi(\mathbb{w}, y_{\mathbb{e}'q}), \Psi(\mathbb{e}', x_q)) < \Delta/3. \quad (7)$$

Since $\{y_{\mathbb{c}} : \mathbb{c} \in \mathbb{C}\}$ is a ρ -pseudo-orbit for Ψ , it follows from (5)–(7) that

$$\begin{aligned} \mathbb{d}(\Psi(\mathbb{e}, x_{\mathbb{w}q}), \Psi(\mathbb{w}, \Psi(\mathbb{e}, x_q))) &\leq \mathbb{d}(\Psi(\mathbb{e}, x_{\mathbb{w}q}), y_{\mathbb{w}\mathbb{e}'q}) + \mathbb{d}(y_{\mathbb{w}\mathbb{e}'q}, \Psi(\mathbb{w}, y_{\mathbb{e}'q})) + \mathbb{d}(\Psi(\mathbb{w}, y_{\mathbb{e}'q}), \Psi(\mathbb{w}, \Psi(\mathbb{e}, x_q))) \\ &\leq \varepsilon + \rho + \Delta/3 < \Delta. \end{aligned}$$

Because the expansivity of $\Psi|_{\mathbb{E}}$ on $\mathbb{M}$, we infer

$$x_{\mathbb{w}q} = \Psi(\mathbb{w}, x_q), \mathbb{w} \in \mathbb{W}, q \in \mathbb{C}.$$

Because $\mathbb{W}$ is a generating set for $\mathbb{C}$, then $x_q = \Psi(q, x_e)$, for all $q \in \mathbb{C}$, and by (4) x_e satisfies inequalities (1).

Hence an action Ψ has shadowing property.

To prove C3, we must prove an action Ψ has inverse shadowing property, let $\rho' = \rho'(\delta) < \Delta/3$ for this δ from the definition of inverse shadowing for $\Psi|_{\mathbb{E}}$. Let Γ be a ρ' -method for Ψ . Fix an arbitrary point $p \in \mathbb{S}$ and an element $\mathbb{c} \in \mathbb{C}$. By inequalities (2) we get that the following

$$\mathbb{d}(\gamma_{\mathbb{w}\mathbb{e}\mathbb{c}}(x), \Psi(\mathbb{w}, \gamma_{\mathbb{e}\mathbb{c}}(x))) < \rho' \quad (8)$$

are satisfied (for any $\mathbb{w} \in \mathbb{W}$ and $\mathbb{e} \in \mathbb{E}$), that is the set

$$\Gamma|_{\mathbb{E}}(\mathbb{c}) = \{\gamma_{\mathbb{e}\mathbb{c}} : \mathbb{e} \in \mathbb{E}\}$$

is a ρ' -method for $\Psi|_{\mathbb{E}}$ (Note that (8) are satisfied for any $\mathbb{e} \in \mathbb{E}$ and $\mathbb{w} \in \mathbb{W}$ not just for $\mathbb{e} \in \mathbb{E}$ and $\mathbb{w} \in \mathbb{W}_{\mathbb{E}}$, as its application follows.). By the definitions of inverse shadowing for $\Psi|_{\mathbb{E}}$, there exists a non-empty set $Z(\mathbb{c}, p)$ of $\mathbb{S}$ such that

$$\left\{ \begin{array}{l} \gamma_{\mathbb{e}\mathbb{c}}(z) : \mathbb{e} \in \mathbb{E}, z \in \\ Z(\mathbb{c}, p) \end{array} \right\} \quad (9)$$

satisfying ρ' -method.

(we shadow the true-orbit of $\Psi(c, p)$ with respect to the action of $\mathbb{E}$), which means that

$$d(\gamma_{e\mathbb{C}}(z), \Psi(e, \Psi(c, p))) < \delta, e \in \mathbb{E}, z \in Z(c, p). \quad 10$$

Now, we take an arbitrary $w \in \mathbb{W}$. Let us take $e \in \mathbb{E}$ and $z \in Z(c, p)$ and estimate

$$d(\gamma_{ew\mathbb{C}}(z), \Psi(ew\mathbb{C}, p)).$$

Since $\mathbb{E}$ is a normal subgroup of $\mathbb{C}$, there is an element $e' \in \mathbb{E}$ such that $we' = ew$.

Then,

$$\begin{aligned} d(\gamma_{ew\mathbb{C}}(z), \Psi(ew\mathbb{C}, p)) &= d(\gamma_{we'\mathbb{C}}(z), \Psi(ew\mathbb{C}, p)) \leq \\ &\leq d(\gamma_{we'\mathbb{C}}(z), \Psi(w, \gamma_{e'\mathbb{C}}(z))) + d(\Psi(w, \gamma_{e'\mathbb{C}}(z)), \Psi(w, \Psi(e'\mathbb{C}, p))) \leq \\ &\leq \rho' + \Delta/3 < \Delta. \text{ (By (9), (10) (with } e = e') \text{ and (3)).} \end{aligned}$$

Note that $\Psi(w, (e'\mathbb{C}, p)) = \Psi(we'\mathbb{C}, p) = \Psi(ew\mathbb{C}, p)$.

By the definitions of inverse shadowing for $\Psi|_{\mathbb{E}}$,

$$d(\gamma_{ew\mathbb{C}}(z), \Psi(ew\mathbb{C}, p)) < \delta, \text{ for } e \in \mathbb{E}.$$

This shows that $Z(c, p) \subset Z(w\mathbb{C}, p)$. But the above reasoning is symmetric, and we conclude that $Z(w\mathbb{C}, p) \subset Z(c, p)$. Hence, the set $Z(c, p)$ is, in fact, independent of c (let us denote this set by $Z(p)$).

Then, we can take $e = e$ in (10) to conclude that

$$d(\gamma_{\mathbb{C}}(z), \Psi(c, p)) < \delta$$

for any $c \in \mathbb{C}$, which gives us the definitions of inverse shadowing for Ψ .

The proof of C4 (Expansivity of Ψ) is trivial. ■

Lemma 2: If Ψ is an action of a nilpotent group $\mathbb{C}$ of class n , $c \in \mathbb{C}$, and $\Psi|_c$ satisfies (B.C.M, N), then also Ψ satisfies (B.C.M, N). Which clear implies that Ψ has bi-shadowing property.

Proof: By induction:

We take $n = 1$, therefore $\mathbb{C}$ is abelian group; and we infer, $\langle c \rangle$ is a normal subgroup of $\mathbb{C}$, and Therefore, we conclude the above statement from result 1(after compensation $\mathbb{E} = \langle c \rangle$).

If we take $n > 1$ and suppose that the lemma 2 statement proved for nilpotent groups when the class not exceed $n - 1$.

We will need to use the helper text below to complete the proof using the same assumptions in Lemma 2.

Lemma 3 [5]: Let $Q = [C, C]$, let $c \in C$, and let $\mathcal{M} = \langle Q, c \rangle$ (i.e. $\mathcal{M}$ is the minimal subgroup of C containing Q and c), then

1. $\mathcal{M}$ is a normal subgroup of a group C .
2. $\mathcal{M}$ is a nilpotent group of class not exceed $n - 1$.

Now we continue to proof Lemma 2.

Clear that any normal subgroup of a finite generating nilpotent group is finite generating. Therefore, $\mathcal{M}$ is a finite generating nilpotent group of class not exceed $n - 1$, $c \in \mathcal{M}$, and $\Psi|_{\langle c \rangle}$ satisfy the conditions of (B.C.M, N).

Now our induction assumption implies that $\Psi|_{\mathcal{M}}$ satisfy the conditions of (B.C.M, N). And by the Lemma 3:1 and Lemma 1 we get that Ψ satisfy the conditions of (B.C.M, N). ■

Below we complete the proof of main Theorem (RBST)

Proof: where the group C is a virtually nilpotent, we can find a normal nilpotent subgroup $\mathbb{E}$ of finite index of group C . Let $c \in C$ that satisfy the conditions of (B.C.M, N). Because that $\mathbb{E}$ has finite index, we can there exist $k > 0$ such that $c^k \in \mathbb{E}$. And the action $\Psi|_{\langle c \rangle}$ satisfy the conditions of (B.C.M, N); hence, the action $\Psi|_{\langle c^k \rangle}$ also satisfy the conditions of (B.C.M, N). By Lemma 2, $\Psi|_{\mathbb{E}}$ satisfies (B.C.M, N). Thus, by apply Lemma 1 to C and $\mathbb{E}$, we complete the proof. ■

Conclusion

we have proven that the action Ψ satisfies the (B. C. M, N) property by verifying conditions C1, C2, C3, and C4 in Definition 2.9. We showed that Ψ has the shadowing property through a series of inequalities and used the expansivity and uniqueness arguments to establish it. Additionally, we demonstrated the inverse shadowing property, proving that the action satisfies the required conditions using a ρ' -method. Expansivity of Ψ was also confirmed. Finally, Lemma 2 and Lemma 3 helped us extend the result to nilpotent and virtually nilpotent groups, completing the proof of the main theorem.

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