

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

Maths 2024; 9(6): 211-220

© 2024 Stats & Maths

<https://www.mathsjournal.com>

Received: 29-10-2024

Accepted: 30-11-2024

Samer Thaaban Abaas AlshibleyDepartment of Mathematics,
Faculty of Education, University
of Kufa, Najaf, Iraq**Ahmed Hameed Olewi**Department of Mathematics,
Faculty of Education, University
of Kufa, Najaf, Iraq**Ahmed Jebur Ali**Department of Mathematics,
Faculty of Education, University
of Kufa, Najaf, Iraq

Applications of fuzzy Kashuri-Fundo transforms

Samer Thaaban Abaas Alshibley, Ahmed Hameed Olewi and Ahmed Jebur Ali

DOI: <https://doi.org/10.22271/math.2024.v9.i6c.1923>

Abstract

The fuzzy Kashuri-Fundo transforms is presented in this study. Properties and related theorems are proved. Examples from everyday life are provided to illustrate the method of this fuzzy transformation in differential equations. This paper demonstrates how this intriguing fuzzy transformation may be used to solve second-order fuzzy differential equations with constant coefficients under generalized Hukuhara differentiability. Furthermore, important theorems are provided to clarify the characteristics of the fuzzy Kashuri-Fundo transform, and the proposed approach is verified using two real-world cases.

Keywords: Second-order fuzzy differential equations, first-order fuzzy differential equations, fuzzy number, fuzzy Kashuri-Fundo transforms, strongly generalized differentiable

1. Introduction

The theory of fuzzy differential equations has gained a lot of interest lately and has been expanding quickly. Numerous writers conducted extensive research on it (see ^[1-4]). Other academics have created some numerical approaches and algorithms to deal with the solutions of this type of problem (see ^[5-8]). A new approach to solving fuzzy linear differential equations was put out by Allahviranloo *et al.* in ^[7]. It was constructed using the equivalent integral forms of the original problems and assumed highly generalized differentiability. However, their major finding was formal and unproven, and their approach was restricted to solving only fuzzy linear differential equations with crisp constant coefficients. We devised and expanded this operator approach in 2015 (see ^[9]) to solve several first-order fuzzy linear differential equations with variable coefficients, motivated by their work. Additionally, we provided the solution to the general formula along with the required proofs. Prior to 2021, Alshibley ^[9] employed the fuzzy Aboodh transform in an analytic solution approach for a few first-order fuzzy differential equations under the highly generalized differentiability. The authors' primary finding was the relationship between a fuzzy function's first derivative and its fuzzy Kashuri-Fundo transformations. They provided two numerical examples, both of which are first-order fuzzy differential equations, to demonstrate the effectiveness of the approach. This work aims to solve second-order fuzzy differential equations under strongly generalized differentiability by extending their main result and developing their method by establishing the relationship between a function's second derivative and its fuzzy Kashuri-Fundo transforms.

2. Fundamental Preliminaries

The following fundamental concepts and theorems pertaining to our work in this field are included for completeness's sake.

2.1 Definition ^[10]

The mapping $B: \mathbb{R} \rightarrow [0, 1]$ is said to be fuzzy number if it fulfills the following:

1. B is upper semi-continuous.
2. B is fuzzy convex, i. e., $B(\gamma\chi + (1 - \gamma)\eta) \geq \min\{B(\chi), B(\eta)\}$, for all $\chi, \eta \in \mathbb{R}$ and $\gamma \in [0, 1]$.
3. B is normal i. e., $\exists \chi_0 \in \mathbb{R}$ for which $B(\chi_0) = 1$.

Corresponding Author:**Samer Thaaban Abaas Alshibley**Department of Mathematics,
Faculty of Education, University
of Kufa, Najaf, Iraq

4. $\text{supp}(\mathbb{B}) = \{ \chi \in \mathbb{R}; \mathbb{B}(\chi) > 0 \}$, and $cl(\text{Supp}(\mathbb{B}))$ is compact.

Let ϖ be the set of all fuzzy number on $\mathbb{R}$. The γ -level set of a fuzzy number $\mathbb{B} \in \varpi$, $0 \leq \gamma \leq 1$ denoted by $[\mathbb{B}]_\gamma$ is defined as

$$[\mathbb{B}]_\gamma = \begin{cases} \{ \chi \in \mathbb{R}, \mathbb{B}(\chi) \geq \gamma \}, & \text{if } 0 \leq \gamma \leq 1 \\ cl(\text{Supp}(\mathbb{B})) & \text{if } \gamma = 0 \end{cases}$$

Done $[\mathbb{B}]_\gamma = [\underline{\mathbb{B}}(\gamma), \overline{\mathbb{B}}(\gamma)]$, so the γ -level set $[\mathbb{B}]_\gamma$ is a bounded and closed interval for all $\gamma \in [0, 1]$.

Zadeh's extension principle states that the operation of addition on ϖ is given by

$$(\mathbb{B} \oplus \Omega)(\chi) = \sup_{t \in \mathbb{R}} \min\{\mathbb{B}(t), \Omega(\chi - t)\}, \chi \in \mathbb{R}$$

and a fuzzy number's scalar multiplication is provided by

$$(\rho \odot \mathbb{B})(\chi) = \begin{cases} \mathbb{B}(\frac{\chi}{\rho}), & \text{if } \rho > 0 \\ \hat{0} & \text{if } \rho = 0 \end{cases} \text{ where } \hat{0} \in \varpi$$

The following characteristics are widely acknowledged to be true at all levels:

$$[\mathbb{B} \oplus \Omega]_\gamma = [\mathbb{B}]_\gamma + [\Omega]_\gamma, [\rho \odot \mathbb{B}]_\gamma = \rho [\mathbb{B}]_\gamma.$$

2.2 Definition ^[11]

A pair that is sorted parametrically is a fuzzy number, $(\underline{\mathbb{B}}, \overline{\mathbb{B}})$ of functions $\overline{\mathbb{B}}(\gamma), \underline{\mathbb{B}}(\gamma), \gamma \in [0, 1]$, which fulfills:

1. $\underline{\mathbb{B}}(\gamma)$ is a continuous function with a right function of 0 and a left function of (0,1] that is non-decreasing.
2. $\overline{\mathbb{B}}(\gamma)$ is a bounded, non-increasing function with 0 continuous right and (0,1] continuous left.
3. $\underline{\mathbb{B}}(\gamma) \leq \overline{\mathbb{B}}(\gamma), \gamma \in [0, 1]$.

For arbitrary $\mathbb{B} = (\underline{\mathbb{B}}(\gamma), \overline{\mathbb{B}}(\gamma)), \mathbb{B} = (\underline{\mathbb{B}}(\gamma), \overline{\mathbb{B}}(\gamma)), 0 \leq \gamma \leq 1$ and $\rho > 0$ we define:

1. Addition $\mathbb{B} \oplus \Omega = (\underline{\mathbb{B}}(\gamma) + \underline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma) + \overline{\Omega}(\gamma))$.
2. Subtraction $\mathbb{B} \ominus_h \Omega = (\underline{\mathbb{B}}(\gamma) - \overline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma) - \underline{\Omega}(\gamma))$.
3. Multiplication $\mathbb{B} \odot \mathbb{B} =$
 1. $(\min\{\underline{\mathbb{B}}(\gamma)\overline{\Omega}(\gamma), \underline{\mathbb{B}}(\gamma)\underline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma)\overline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma)\underline{\Omega}(\gamma)\}, \max\{\underline{\mathbb{B}}(\gamma)\overline{\Omega}(\gamma), \underline{\mathbb{B}}(\gamma)\underline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma)\overline{\Omega}(\gamma), \overline{\mathbb{B}}(\gamma)\underline{\Omega}(\gamma)\})$
4. Scalar multiplication $\rho \odot \mathbb{B} = \begin{cases} (\rho \underline{\mathbb{B}}, \rho \overline{\mathbb{B}}) & \rho \geq 0, \\ (\rho \overline{\mathbb{B}}, \rho \underline{\mathbb{B}}) & \rho < 0. \end{cases}$ If $\rho = 1$ then $\rho \odot \mathbb{B} = -\mathbb{B}$

2.3 Definition ^[11]

Let $\mathbb{B}$ and Ω are fuzzy numbers, the Hausdorff distance between two fuzzy numbers is provided by:

$$\Gamma: \varpi \times \varpi \rightarrow [0, +\infty]$$

$$\Gamma(\mathbb{B}, \Omega) = \sup_{\gamma \in [0, 1]} \max\{|\underline{\mathbb{B}}(\gamma) - \underline{\Omega}(\gamma)|, |\overline{\mathbb{B}}(\gamma) - \overline{\Omega}(\gamma)|\},$$

Where $\mathbb{B} = (\underline{\mathbb{B}}(\gamma), \overline{\mathbb{B}}(\gamma)), \mathbb{B} = (\underline{\mathbb{B}}(\gamma), \overline{\mathbb{B}}(\gamma)) \subset \mathbb{R}$ and following properties are well known:

1. $\Gamma(\mathbb{B} \oplus \mathbb{A}, \Omega \oplus \mathbb{B}) = \Gamma(\mathbb{B}, \Omega), \forall \mathbb{B}, \Omega, \mathbb{A} \in \varpi$.
2. $\Gamma(\rho \odot \mathbb{B}, \rho \odot \Omega) = |\rho| \Gamma(\mathbb{B}, \Omega), \forall \mathbb{B}, \Omega \in \varpi, \rho \in \mathbb{R}$.
3. $\Gamma(\mathbb{B} \oplus \Omega, \mathbb{A} \oplus \mathbb{B}) \leq \Gamma(\mathbb{B}, \Omega) + \Gamma(\mathbb{A}, \mathbb{B}), \forall \mathbb{B}, \Omega, \mathbb{A}, \mathbb{B} \in \varpi$.
4. (Γ, ϖ) is a complete metric space.

2.4 Definition ^[11]

Let $\Upsilon: \mathbb{R} \rightarrow \varpi$ be a function with fuzzy values. Assuming a random fixed point $\chi_0 \in \mathbb{R}$ and $\epsilon > 0$ there exists $\delta > 0$ such that $|\chi - \chi_0| < \delta \Rightarrow \Gamma(\Upsilon(\chi), \Upsilon(\chi_0)) < \epsilon$, $\mathbb{B}$ is said to be a continuous fuzzy-valued function.

2.5 Definition ^[12]

A mapping $\Upsilon: \mathbb{R} \times \varpi \rightarrow \varpi$ is referred to as continuous at one point $(\tau_0, \chi_0) \in \mathbb{R} \times \varpi$ supplied for any fixed $\gamma_0 \in [0, 1]$ and arbitrary $\epsilon > 0$ there exists an $\delta(\epsilon, \gamma)$ such that $\Gamma([\Upsilon(\tau, \chi)]_\gamma, [\Upsilon(\tau_0, \chi_0)]_\gamma) < \epsilon$ whenever $|\tau - \tau_0| < \delta(\epsilon, \gamma)$ and $\Gamma([\chi]_\gamma, [\chi_0]_\gamma) < \delta(\epsilon, \gamma)$ for all $\tau \in \mathbb{R}, \chi \in \varpi$

2.5.1 Theorem ^[13]

Assume $\Upsilon(\chi)$ function with fuzzy values on $[e, \infty)$ and it is embodied by $(\underline{\Upsilon}(\chi, \gamma), \overline{\Upsilon}(\chi, \gamma))$. For any fixed $\gamma \in [0, 1]$, let $\underline{\Upsilon}(\chi, \gamma)$ and $\overline{\Upsilon}(\chi, \gamma)$ are Riemann-integrals on $[p, q]$. For every $q \geq p$, if there are two positive functions $\underline{\theta}(\gamma)$ and $\overline{\theta}(\gamma)$ such that

$\int_p^q |\underline{Y}(\chi, \mathbb{Y})| d\chi \leq \underline{\Theta}(\mathbb{Y})$ and $\int_p^q |\overline{Y}(\chi, \mathbb{Y})| d\chi \leq \overline{\Theta}(\mathbb{Y})$, then the fuzzy number is the improper fuzzy Riemann-integrable, and $Y(\chi)$ is said to be improper fuzzy Riemann-integrable on $[p, \infty]$, i.e.

$$\int_p^\infty Y(\chi) d\chi = \left[\int_p^\infty (\underline{Y}(\chi, \mathbb{Y})) d\chi, \int_p^\infty (\overline{Y}(\chi, \mathbb{Y})) d\chi \right]$$

2.6 Definition ^[12]

Assume $\psi, \lambda \in \mathcal{W}$. There is $\beta \in \mathcal{W}$ such that $\psi = \lambda \oplus \beta$ then β is known the H-differential of ψ and λ and it is represented by $\psi \ominus \lambda$. In this paper, the sign " $\ominus$ " always stands for H-difference, and also note that

$$\ominus \neq \ominus_h \text{ and } \psi \ominus \lambda \neq \psi + (-1)\lambda$$

2.7 Definition ^[14]

A function $Y: (p, q) \rightarrow \mathcal{W}$ and $\chi_0 \in (p, q)$. We say that Y is strongly generalized differentiable at χ_0 if such an element exists $(\chi_0) \in \mathcal{W}$, such that:

i. $\forall \varepsilon > 0$ that is adequately little, there are $Y(\chi_0 + \varepsilon) \ominus Y(\chi_0), Y(\chi_0) \ominus Y(\chi_0 - \varepsilon)$,

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0 + \varepsilon) \ominus Y(\chi_0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0) \ominus Y(\chi_0 - \varepsilon)}{\varepsilon} = Y'(\chi_0)$$

Or

ii. $\forall \varepsilon > 0$ that is adequately little, there are $Y(\chi_0) \ominus Y(\chi_0 + \varepsilon), Y(\chi_0 - \varepsilon) \ominus Y(\chi_0)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0) \ominus Y(\chi_0 + \varepsilon)}{-\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0 - \varepsilon) \ominus Y(\chi_0)}{-\varepsilon} = Y'(\chi_0)$$

Or

iii. $\forall \varepsilon > 0$ that is adequately little, there are $Y(\chi_0 + \varepsilon) \ominus Y(\chi_0), Y(\chi_0 - \varepsilon) \ominus Y(\chi_0)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0 + \varepsilon) \ominus Y(\chi_0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0 - \varepsilon) \ominus Y(\chi_0)}{-\varepsilon} = Y'(\chi_0)$$

Or

iv. $\forall \varepsilon > 0$ that is adequately little, there are $Y(\chi_0) \ominus Y(\chi_0 + \varepsilon), Y(\chi_0) \ominus Y(\chi_0 - \varepsilon)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0) \ominus Y(\chi_0 + \varepsilon)}{-\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y(\chi_0) \ominus Y(\chi_0 - \varepsilon)}{\varepsilon} = Y'(\chi_0).$$

2.9 Theorem ^[15]

Let $Y(\chi): \mathbb{R} \rightarrow \mathcal{W}$ be a function and represents $Y(\chi) = ((\underline{Y}(\chi, \mathbb{Y}), \overline{Y}(\chi, \mathbb{Y})))$ in every instance for $\mathbb{Y} \in [0, 1]$. Then:

1. If $Y(\chi)$ is differentiable form i, then $(\underline{Y}(\chi, \mathbb{Y})$ and $\overline{Y}(\chi, \mathbb{Y})$ are differentiable functions and $Y'(\chi) = (\underline{Y}'(\chi, \mathbb{Y}), \overline{Y}'(\chi, \mathbb{Y}))$.
2. If $Y(\chi)$ is differentiable form ii, then $(\underline{Y}(\chi, \mathbb{Y})$ and $\overline{Y}(\chi, \mathbb{Y})$ are differentiable functions and $Y'(\chi) = (\overline{Y}'(\chi, \mathbb{Y}), \underline{Y}'(\chi, \mathbb{Y}))$.

2.10 Definition ^[16]

A function $Y: (p, q) \rightarrow \mathcal{W}$. We're saying that a mapping Y is strongly generalized differentiable at χ_0 if an element $Y^{(2)}(\chi_0) \in \mathcal{W}$ exists, such that:

i. $\forall \varepsilon > 0$ that is sufficiently small, there exist $Y'(\chi_0 + \varepsilon) \ominus Y'(\chi_0), Y'(\chi_0) \ominus Y'(\chi_0 - \varepsilon)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y'(\chi_0 + \varepsilon) \ominus Y'(\chi_0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y'(\chi_0) \ominus Y'(\chi_0 - \varepsilon)}{\varepsilon} = Y^{(2)}(\chi_0)$$

Or

ii. $\forall \varepsilon > 0$ that is sufficiently small, there exist $Y'(\chi_0) \ominus Y'(\chi_0 + \varepsilon), Y'(\chi_0 - \varepsilon) \ominus Y'(\chi_0)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{Y'(\chi_0) \ominus Y'(\chi_0 + \varepsilon)}{-\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{Y'(\chi_0 - \varepsilon) \ominus Y'(\chi_0)}{-\varepsilon} = Y^{(2)}(\chi_0)$$

Or

iii. $\forall \varepsilon > 0$ that is sufficiently small, there exist $\Upsilon'(\chi_0 + \varepsilon) \ominus \Upsilon'(\chi_0), \Upsilon'(\chi_0 - \varepsilon) \ominus \Upsilon'(\chi_0)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{\Upsilon'(\chi_0 + \varepsilon) \ominus \Upsilon'(\chi_0)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\Upsilon'(\chi_0 - \varepsilon) \ominus \Upsilon'(\chi_0)}{-\varepsilon} = \Upsilon^{(2)}(\chi_0)$$

Or

iv. $\forall \varepsilon > 0$ that is sufficiently small, there exist $\Upsilon'(\chi_0) \ominus \Upsilon'(\chi_0 + \varepsilon), \Upsilon'(\chi_0) \ominus \Upsilon'(\chi_0 - \varepsilon)$

$$\text{Where } \lim_{\varepsilon \rightarrow 0} \frac{\Upsilon'(\chi_0) \ominus \Upsilon'(\chi_0 + \varepsilon)}{-\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{\Upsilon'(\chi_0) \ominus \Upsilon'(\chi_0 - \varepsilon)}{\varepsilon} = \Upsilon^{(2)}(\chi_0).$$

2.11 Theorem ^[17]

Let $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are two differentiable fuzzy-valued functions. Denoted that Υ -cut is fuzzy valued function representation $\Upsilon(\chi)$ with $\Upsilon(\chi) = [\underline{\Upsilon}(\chi, \Upsilon), \bar{\Upsilon}(\chi, \Upsilon)]$, then:

1. Let $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are (i)-differentiable, or let $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are (ii)-differentiable then $\underline{\Upsilon}(\chi, \Upsilon)$ and $\bar{\Upsilon}(\chi, \Upsilon)$ have derivatives in first and second order such that

$$\Upsilon^{(2)}(\chi) = [\underline{\Upsilon}^{(2)}(\chi, \Upsilon), \bar{\Upsilon}^{(2)}(\chi, \Upsilon)].$$

2. Let $\Upsilon(\chi)$ be (i)-differentiable and $\Upsilon'(\chi)$ be (ii)-differentiable, or let $\Upsilon(\chi)$ be (ii)-differentiable and $\Upsilon'(\chi)$ be (i) differentiable then $\underline{\Upsilon}(\chi, \Upsilon)$ and $\bar{\Upsilon}(\chi, \Upsilon)$ have derivatives in first and second order such that $\Upsilon^{(2)}(\chi) = [\bar{\Upsilon}^{(2)}(\chi, \Upsilon), \underline{\Upsilon}^{(2)}(\chi, \Upsilon)]$.

3. Fuzzy Kashuri-Fundo Transform

The fuzzy start and boundary value issues related to fuzzy differential equations are resolved by the fuzzy Kashuri-Fundo transform method. Fuzzy Kashuri-Fundo transformations simplify a fuzzy differential equation by turning it into an algebraic problem. Operational calculus, a crucial branch of practical mathematics, is the transition from calculus operations to algebraic operations on transformations.

Artion Kashuri and Akli Fundo.

3.1 Kashuri-Fundo Integral Transform ^[18]

The Kashuri-Fundo Transform is an integral transform that was proposed by Kashuri and Fundo. The function defines this transform $\Upsilon(\chi)$ as:

$$\mathbb{K}[\Upsilon(\chi)] = \frac{1}{r} \int_0^\infty \Upsilon(\chi) e^{-\frac{1}{r^2}\chi} d\chi = \mathbb{Q}(r), \text{ Where } \chi \geq 0, -\mu_1 \leq r \leq \mu_1 \text{ and in the function } \Upsilon, \text{ the variable } r \text{ is used as a factor to the variable } \chi$$

3.2 Definition

Let $\Upsilon(\chi)$ be a fuzzy-valued continuous function. Suppose that $\frac{1}{r} \Upsilon(\chi) \odot e^{-\frac{1}{r^2}\chi}$ is an inappropriate fuzzy Integrable at Rimann on $[0, \infty)$, then $\frac{1}{r} \int_0^\infty \Upsilon(\chi) e^{-\frac{1}{r^2}\chi} d\chi = \mathbb{Q}(r)$ is being called fuzzy Kashuri-Fundo transform and is known as

$$\hat{\mathbb{K}}[\Upsilon(\chi)] = \frac{1}{r} \int_0^\infty \Upsilon(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi, (r > 0 \text{ and integer})$$

$$\frac{1}{r} \int_0^\infty \Upsilon(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi = \left(\frac{1}{r} \int_0^\infty \underline{\Upsilon}(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi, \frac{1}{r} \int_0^\infty \bar{\Upsilon}(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi \right).$$

Using the definition of classical Kashuri-Fundo transform, to get:

$$\mathbb{K}[\underline{\Upsilon}(\chi, \Upsilon)] = \frac{1}{r} \int_0^\infty \underline{\Upsilon}(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi \text{ and } \mathbb{K}[\bar{\Upsilon}(\chi, \Upsilon)] = \frac{1}{r} \int_0^\infty \bar{\Upsilon}(\chi) \odot e^{-\frac{1}{r^2}\chi} d\chi, \text{ then}$$

$$\hat{\mathbb{K}}[\Upsilon(\chi)] = (\mathbb{K}[\underline{\Upsilon}(\chi, \Upsilon)], \mathbb{K}[\bar{\Upsilon}(\chi, \Upsilon)])$$

3.3 Theorem

Let $\Upsilon(\chi), \mathbb{Y}(\chi)$ be continuous fuzzy-valued functions, $\mathcal{M}_1$ and $\mathcal{M}_2$ are constants, then

$$(1). \hat{\mathbb{K}}[\mathcal{M}_1 \odot \Upsilon(\chi)] = \mathcal{M}_1 \odot \hat{\mathbb{K}}[\Upsilon(\chi)].$$

$$(2). \hat{\mathbb{K}}[(\mathcal{M}_1 \odot \Upsilon(\chi)) \oplus (\mathcal{M}_2 \odot \mathbb{Y}(\chi))] = (\mathcal{M}_1 \odot \hat{\mathbb{K}}[\Upsilon(\chi)]) \oplus (\mathcal{M}_2 \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)])$$

Proof

$$\begin{aligned}\hat{\mathbb{K}}[\mathcal{M}_1 \odot \mathbb{Y}(\chi)] &= (\mathbb{K}[\mathcal{M}_1 \underline{\mathbb{Y}}(\chi, \mathbb{Y})], \mathbb{K}[\mathcal{M}_1 \overline{\mathbb{Y}}(\chi, \mathbb{Y})]) = \left(\frac{1}{r} \int_0^\infty \mathcal{M}_1 \underline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi, \frac{1}{r} \int_0^\infty \mathcal{M}_1 \overline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi \right) \\ &= \left(\mathcal{M}_1 \frac{1}{r} \int_0^\infty \underline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi, \mathcal{M}_1 \frac{1}{r} \int_0^\infty \overline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi \right) \\ &= \mathcal{M}_1 \left(\frac{1}{r} \int_0^\infty \underline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi, \frac{1}{r} \int_0^\infty \overline{\mathbb{Y}}(\chi) \odot e^{-\frac{1}{r^2} \chi} d\chi \right) = \mathcal{M}_1 (\mathbb{K}[\underline{\mathbb{Y}}(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbb{Y}}(\chi, \mathbb{Y})]) = \mathcal{M}_1 \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)]\end{aligned}$$

(2). Suppose $\mathbb{Y}(\chi) = (\underline{\mathbb{Y}}(\chi, \mathbb{Y}), \overline{\mathbb{Y}}(\chi, \mathbb{Y}))$ and $\mathbb{Z}(\chi) = (\underline{\mathbb{Z}}(\chi, \mathbb{Y}), \overline{\mathbb{Z}}(\chi, \mathbb{Y}))$

$$\begin{aligned}\hat{\mathbb{K}}[(\mathcal{M}_1 \odot \mathbb{Y}(\chi)) \oplus (\mathcal{M}_2 \odot \mathbb{Z}(\chi))] &= (\mathbb{K}[\mathcal{M}_1 \underline{\mathbb{Y}}(\chi, \mathbb{Y}) + \mathcal{M}_2 \underline{\mathbb{Z}}(\chi, \mathbb{Y})], \mathbb{K}[\mathcal{M}_1 \overline{\mathbb{Y}}(\chi, \mathbb{Y}) + \mathcal{M}_2 \overline{\mathbb{Z}}(\chi, \mathbb{Y})]) = \left(\frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} (\mathcal{M}_1 \underline{\mathbb{Y}}(\chi, \mathbb{Y}) + \mathcal{M}_2 \underline{\mathbb{Z}}(\chi, \mathbb{Y})) d\chi, \frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} (\mathcal{M}_1 \overline{\mathbb{Y}}(\chi, \mathbb{Y}) + \mathcal{M}_2 \overline{\mathbb{Z}}(\chi, \mathbb{Y})) d\chi \right) \\ &= \left(\frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} \mathcal{M}_1 \underline{\mathbb{Y}}(\chi, \mathbb{Y}) d\chi, \frac{1}{r} \int_0^\infty \mathcal{M}_1 \overline{\mathbb{Y}}(\chi, \mathbb{Y}) e^{-\frac{1}{r^2} \chi} d\chi \right) + \left(\frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} \mathcal{M}_2 \underline{\mathbb{Z}}(\chi, \mathbb{Y}) d\chi, \frac{1}{r} \int_0^\infty \mathcal{M}_2 \overline{\mathbb{Z}}(\chi, \mathbb{Y}) e^{-\frac{1}{r^2} \chi} d\chi \right) \\ &= \mathcal{M}_1 \left(\frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} \underline{\mathbb{Y}}(\chi, \mathbb{Y}) d\chi, \frac{1}{r} \int_0^\infty \overline{\mathbb{Y}}(\chi, \mathbb{Y}) e^{-\frac{1}{r^2} \chi} d\chi \right) \\ &\quad + \mathcal{M}_2 \left(\frac{1}{r} \int_0^\infty e^{-\frac{1}{r^2} \chi} \underline{\mathbb{Z}}(\chi, \mathbb{Y}) d\chi, \frac{1}{r} \int_0^\infty \overline{\mathbb{Z}}(\chi, \mathbb{Y}) e^{-\frac{1}{r^2} \chi} d\chi \right) \\ &= \mathcal{M}_1 (\mathbb{K}[\underline{\mathbb{Y}}(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbb{Y}}(\chi, \mathbb{Y})]) + \mathcal{M}_2 (\mathbb{K}[\underline{\mathbb{Z}}(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbb{Z}}(\chi, \mathbb{Y})]) = (\mathcal{M}_1 \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)]) \oplus (\mathcal{M}_2 \odot \hat{\mathbb{K}}[\mathbb{Z}(\chi)])\end{aligned}$$

4. Fuzzy Kashuri-Fundo Transform for First -Order Fuzzy Differential Equation

The fuzzy Kashuri-Fundo transform of the derivative of a first order under generalized H-differentiability must be analyzed for high-order fuzzy differential equations.

4.1 Theorem

Let $\mathbb{Y}(\chi)$ is the primitive of $\mathbb{Y}'(\chi)$ on $[0, \infty)$ and $\mathbb{Y}(\chi)$ is a fuzzy-valued function that is integrable, then:

1. $\mathbb{Y}(\chi)$ is (i)-differentiable then $\hat{\mathbb{K}}[\mathbb{Y}'(\chi)] = \frac{1}{r^2} \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)] \ominus \frac{1}{r} \odot \mathbb{Y}(0)$.
2. $\mathbb{Y}(\chi)$ is (ii)-differentiable then $\hat{\mathbb{K}}[\mathbb{Y}'(\chi)] = \left(-\frac{1}{r} \odot \mathbb{Y}(0) \right) \ominus \left(-\frac{1}{r^2} \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)] \right)$

Proof (1)

For a fixed, arbitrary $0 \leq \mathbb{Y} \leq 1$,

$$\frac{1}{r^2} \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)] \ominus \frac{1}{r} \odot \mathbb{Y}(0) = \left(\frac{1}{r^2} \mathbb{K}[\underline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \underline{\mathbb{Y}}(0, \mathbb{Y}), \frac{1}{r^2} \mathbb{K}[\overline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \overline{\mathbb{Y}}(0, \mathbb{Y}) \right)$$

Since

$$\mathbb{K}[\underline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \frac{1}{r^2} \mathbb{K}[\underline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \underline{\mathbb{Y}}(0, \mathbb{Y}), \mathbb{K}[\overline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \frac{1}{r^2} \mathbb{K}[\overline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \overline{\mathbb{Y}}(0, \mathbb{Y}).$$

Since $\mathbb{Y}(\chi)$ is differentiable of form i using Theorem (2.2)

$$\underline{\mathbb{Y}}'(\chi, \mathbb{Y}) = \underline{\mathbb{Y}}'(\chi, \mathbb{Y}), \overline{\mathbb{Y}}'(\chi, \mathbb{Y}) = \overline{\mathbb{Y}}'(\chi, \mathbb{Y})$$

$$\mathbb{K}[\underline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \mathbb{K}[\underline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \frac{1}{r^2} \mathbb{K}[\underline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \underline{\mathbb{Y}}(0, \mathbb{Y})$$

$$\mathbb{K}[\overline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \mathbb{K}[\overline{\mathbb{Y}}'(\chi, \mathbb{Y})] = \frac{1}{r^2} \mathbb{K}[\overline{\mathbb{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^2} \overline{\mathbb{Y}}(0, \mathbb{Y})$$

$$\frac{1}{r^2} \odot \hat{\mathbb{K}}[\mathbb{Y}(\chi)] \ominus \frac{1}{r} \odot \mathbb{Y}(0) = (\mathbb{K}[\underline{\mathbb{Y}}'(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbb{Y}}'(\chi, \mathbb{Y})]) = \hat{\mathbb{K}}[\mathbb{Y}'(\chi)]$$

Proof (2)

$$\left(-\frac{1}{r} \odot Y(0)\right) \ominus \left(-\frac{1}{r^2} \odot \hat{K}[Y(\chi)]\right) = \left(\frac{1}{r^2} K[\bar{Y}(\chi, Y)] - \frac{1}{r} \bar{Y}(0, Y), \frac{1}{r^2} K[Y(\chi, Y)] - \frac{1}{r} Y(0, Y)\right)$$

Since

$$K[Y'(\chi, Y)] = \frac{1}{r^2} K[Y(\chi, Y)] - \frac{1}{r} Y(0, Y), K[\bar{Y}'(\chi, Y)] = \frac{1}{r^2} K[\bar{Y}(\chi, Y)] - \frac{1}{r} \bar{Y}(0, Y).$$

Since $Y(\chi)$ is differentiable of form ii using Theorem (2.2):

$$\underline{Y}'(\chi, Y) = \bar{Y}'(\chi, Y), \bar{Y}'(\chi, Y) = \underline{Y}'(\chi, Y)$$

$$K[\underline{Y}'(\chi, Y)] = K[\bar{Y}'(\chi, Y)] = -\frac{1}{r} \bar{Y}(0, Y) + \frac{1}{r^2} K[\bar{Y}(\chi, Y)]$$

$$K[\bar{Y}'(\chi, Y)] = K[\underline{Y}'(\chi, Y)] = -\frac{1}{r} Y(0, Y) + \frac{1}{r^2} K[Y(\chi, Y)]$$

$$\left(-\frac{1}{r} \odot Y(0)\right) \ominus \left(-\frac{1}{r^2} \odot \hat{K}[Y(\chi)]\right) = (K[\underline{Y}'(\chi, Y)], K[\bar{Y}'(\chi, Y)]) = \hat{K}[Y'(\chi)]$$

4.1 Example: Consider a fuzzy initial value problem:

$$Y'(\chi) = Y(\chi), Y(0, Y) = (Y - 1, 1 - Y), 0 \leq Y \leq 1.$$

Solution

Apply both sides' fuzzy Kashuri-Fundo transforms to get $\hat{K}[Y'(\chi)] = \hat{K}[Y(\chi)]$

Case (1)

$Y(\chi)$ be (i)-differentiable,

$$\hat{K}[Y'(\chi)] = \frac{1}{r^2} \odot \hat{K}[Y(\chi)] \ominus \frac{1}{r} \odot Y(0)$$

Using upper and lower functions, to have

$$\frac{1}{r^2} K[Y(\chi, Y)] - \frac{1}{r} Y(0, Y) = K[\underline{Y}(\chi, Y)], \frac{1}{r^2} K[\bar{Y}(\chi, Y)] - \frac{1}{r} \bar{Y}(0, Y) = K[\bar{Y}(\chi, Y)]$$

$$\left(\frac{1}{r^2} - 1\right) K[\underline{Y}(\chi, Y)] = \frac{1}{r} (Y - 1), \left(\frac{1}{r^2} - 1\right) K[\bar{Y}(\chi, Y)] = \frac{1}{r} (1 - Y)$$

$$K[\underline{Y}(\chi, Y)] = \frac{\frac{1}{r}}{\left(\frac{1}{r^2} - 1\right)} (Y - 1), K[\bar{Y}(\chi, Y)] = \frac{\frac{1}{r}}{\left(\frac{1}{r^2} - 1\right)} (1 - Y)$$

$$\underline{Y}(\chi, Y) = (K)^{-1} \left(\frac{r}{(1 - r^2)} (Y - 1) \right), \bar{Y}(\chi, Y) = (K)^{-1} \left(\frac{r}{(1 - r^2)} (1 - Y) \right)$$

Using inverse Kashuri-Fundo transform, $\underline{Y}(\chi, Y) = (Y - 1) e^{\chi}, \bar{Y}(\chi, Y) = (1 - Y) e^{\chi}$

Case (2)

$Y(\chi)$ be (ii)-differentiable,

$$\hat{K}[Y'(\chi)] = \left(-\frac{1}{r} \odot Y(0)\right) \ominus \left(-\frac{1}{r^2} \odot \hat{K}[Y(\chi)]\right)$$

Using upper and lower functions, to have

$$-\frac{1}{r} \bar{Y}(0, Y) + \frac{1}{r^2} K[\bar{Y}(\chi, Y)] = K[\underline{Y}(\chi, Y)], -\frac{1}{r} Y(0, Y) + \frac{1}{r^2} K[Y(\chi, Y)] = K[\bar{Y}(\chi, Y)]$$

$$-\frac{1}{r}(1-\gamma) + \frac{1}{r^2} \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] = \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)], -\frac{1}{r}(\gamma-1) + \frac{1}{r^2} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] = \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)]$$

$$\frac{1}{r^2} \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] - \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] = \frac{1}{r}(1-\gamma), \frac{1}{r^2} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] - \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] = \frac{1}{r}(\gamma-1)$$

$$\mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] = \frac{r}{(1+r^2)}(1-\gamma), \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] = \frac{r}{(1+r^2)}(\gamma-1)$$

Using inverse Kashuri-Fundo transform, $\underline{\mathbf{Y}}(\chi, \gamma) = (1-\gamma)e^{-\chi}$, $\bar{\mathbf{Y}}(\chi, \gamma) = (1-\gamma)e^{-\chi}$

5. Fuzzy Kashuri-Fundo transform for second-order differential equation

In this part, fuzzy Ordinary differential equations with second order derivative are solved using the fuzzy Kashuri-Fundo transform.

5.1 Theorem

Let $\mathbf{Y}(\chi)e^{-s\chi}$, $\mathbf{Y}'(\chi)e^{-s\chi}$ and $\mathbf{Y}^{(2)}(\chi)e^{-s\chi}$ are continuous and integrable Riemann functions on $[0, \infty)$ so $\mathbf{Y}(\chi)$ is continuous fuzzy valued function. Thus:

a) If $\mathbf{Y}(\chi)$ and $\mathbf{Y}'(\chi)$ are (i)-differentiable, then

$$\mathbb{K}[\mathbf{Y}^{(2)}(\chi)] = \left\{ \frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \ominus \frac{1}{r^3} \odot \mathbf{Y}(0) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0).$$

b) If $\mathbf{Y}(\chi)$ is (i)-differentiable and $\mathbf{Y}'(\chi)$ is (ii)-differentiable, then

$$\mathbb{K}[\mathbf{Y}^{(2)}(\chi)] = \left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ -\frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \ominus \left(-\frac{1}{r^3} \odot \mathbf{Y}(0) \right) \right\}$$

c) If $\mathbf{Y}(\chi)$ is (ii)-differentiable and $\mathbf{Y}'(\chi)$ is (i)-differentiable, then

$$\mathbb{K}[\mathbf{Y}^{(2)}(\chi)] = \left\{ -\frac{1}{r^3} \odot \mathbf{Y}(0) \ominus \left(-\frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \right) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0)$$

d) If $\mathbf{Y}(\chi)$ is (ii)-differentiable and $\mathbf{Y}'(\chi)$ is (ii)-differentiable, then

$$\mathbb{K}[\mathbf{Y}^{(2)}(\chi)] = \left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ \frac{1}{r^3} \odot \mathbf{Y}(0) \ominus \frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \right\}$$

Proof: (a) $\mathbf{Y}(\chi)$ and $\mathbf{Y}'(\chi)$ are (i)-differentiable

$$\left\{ \frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \ominus \frac{1}{r^3} \odot \mathbf{Y}(0) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0) = \left(\frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \odot \underline{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \underline{\mathbf{Y}}'(0, \gamma), \frac{1}{r^4} \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \odot \bar{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \bar{\mathbf{Y}}'(0, \gamma) \right),$$

$$\{s^2 \hat{\mathbf{A}}[\mathbf{Y}(\chi)] \ominus \mathbf{Y}(0)\} \ominus \frac{1}{s} \mathbf{Y}'(0) = \left(s^2 \mathbf{A}[\underline{\mathbf{Y}}(\chi, \gamma)] - \underline{\mathbf{Y}}(0, \gamma) - \frac{1}{s} \underline{\mathbf{Y}}'(0, \gamma), s^2 \mathbf{A}[\bar{\mathbf{Y}}(\chi, \gamma)] - \bar{\mathbf{Y}}(0, \gamma) - \frac{1}{s} \bar{\mathbf{Y}}'(0, \gamma) \right)$$

Since

$$\mathbb{K}[\underline{\mathbf{Y}}^{(2)}(\chi, \gamma)] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \underline{\mathbf{Y}}'(0, \gamma),$$

$$\mathbb{K}[\bar{\mathbf{Y}}^{(2)}(\chi, \gamma)] = \frac{1}{r^4} \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \bar{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \bar{\mathbf{Y}}'(0, \gamma).$$

Since $\mathbf{Y}(\chi)$ and $\mathbf{Y}'(\chi)$ are (i)-differentiable, using Theorem (2.2):

$$\underline{\mathbf{Y}}^{(2)}(\chi, \gamma) = \underline{\mathbf{Y}}^{(2)}(\chi, \gamma), \bar{\mathbf{Y}}^{(2)}(\chi, \gamma) = \bar{\mathbf{Y}}^{(2)}(\chi, \gamma).$$

Since $\mathbf{Y}(\chi)$ is (i)-differentiable using Theorem (2.1),

$$\bar{\mathbf{Y}}'(0, \gamma) = \bar{\mathbf{Y}}'(0, \gamma) \text{ and } \underline{\mathbf{Y}}'(0, \gamma) = \underline{\mathbf{Y}}'(0, \gamma).$$

$$\mathbb{K}[\underline{\mathbf{Y}}^{(2)}(\chi, \gamma)] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \underline{\mathbf{Y}}'(0, \gamma),$$

$$\mathbb{K}[\bar{\mathbf{Y}}^{(2)}(\chi, \gamma)] = \frac{1}{r^4} \mathbb{K}[\bar{\mathbf{Y}}(\chi, \gamma)] - \frac{1}{r^3} \bar{\mathbf{Y}}(0, \gamma) - \frac{1}{r} \bar{\mathbf{Y}}'(0, \gamma).$$

$$\left\{ \frac{1}{r^4} \odot \mathbb{K}[\mathbf{Y}(\chi)] \ominus \frac{1}{r^3} \odot \mathbf{Y}(0) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0) = \left(\mathbb{K}[\underline{\mathbf{Y}}^{(2)}(\chi, \gamma)], \mathbb{K}[\bar{\mathbf{Y}}^{(2)}(\chi, \gamma)] \right) = \mathbb{K}[\mathbf{Y}^{(2)}(\chi)].$$

$$\hat{\mathbb{K}}[\mathbf{Y}^{(2)}(\chi)] = \left\{ \frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \ominus \frac{1}{r^3} \odot \mathbf{Y}(0) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0).$$

(b) $\mathbf{Y}(\chi)$ is (i)-differentiable and $\mathbf{Y}'(\chi)$ is (ii)-differentiable

$$\left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ -\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \ominus \left(-\frac{1}{r^3} \odot \mathbf{Y}(0) \right) \right\} = \left(-\frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}) + \frac{1}{r^4} \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \odot \overline{\mathbf{Y}}(0, \mathbb{Y}), -\frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}) + \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \odot \underline{\mathbf{Y}}(0, \mathbb{Y}) \right),$$

Since

$$\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}),$$

$$\mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \overline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}).$$

Since, $\mathbf{Y}(\chi)$ is (i)-differentiable and $\mathbf{Y}'(\chi)$ is (ii)-differentiable using Theorem (2.2):

$$\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}) = \underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}), \underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}) = \overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}).$$

Since $\mathbf{Y}(\chi)$ is (i)-differentiable using Theorem (2.1) getting,

$$\overline{\mathbf{Y}'}(0, \mathbb{Y}) = \overline{\mathbf{Y}'}(0, \mathbb{Y}) \text{ and } \underline{\mathbf{Y}'}(0, \mathbb{Y}) = \underline{\mathbf{Y}'}(0, \mathbb{Y}).$$

$$\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}),$$

$$\mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \overline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}).$$

$$\left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ -\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \ominus \left(-\frac{1}{r^3} \odot \mathbf{Y}(0) \right) \right\} = \left(\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] \right) = \hat{\mathbb{K}}[\mathbf{Y}^{(2)}(\chi)].$$

$$\hat{\mathbb{K}}[\mathbf{Y}^{(2)}(\chi)] = \left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ -\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \ominus \left(-\frac{1}{r^3} \odot \mathbf{Y}(0) \right) \right\}.$$

(c) $\mathbf{Y}(\chi)$ is (ii)-differentiable and $\mathbf{Y}'(\chi)$ is (i)-differentiable

$$\left\{ -\frac{1}{r^3} \odot \mathbf{Y}(0) \ominus \left(-\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \right) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0) = \left(-\frac{1}{r^3} \odot \overline{\mathbf{Y}}(0, \mathbb{Y}) + \frac{1}{r^4} \odot \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}), -\frac{1}{r^3} \odot \underline{\mathbf{Y}}(0, \mathbb{Y}) + \frac{1}{r^4} \odot \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}) \right),$$

Since

$$\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}),$$

$$\mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \overline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}).$$

Since $\mathbf{Y}(\chi)$ is (ii)-differentiable and $\mathbf{Y}'(\chi)$ is (i)-differentiable using Theorem (2.2):

$$\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}) = \underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}), \underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}) = \overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y}).$$

Since $\mathbf{Y}(\chi)$ is (ii)-differentiable using Theorem (2.1) getting,

$$\overline{\mathbf{Y}'}(0, \mathbb{Y}) = \underline{\mathbf{Y}'}(0, \mathbb{Y}) \text{ and } \underline{\mathbf{Y}'}(0, \mathbb{Y}) = \overline{\mathbf{Y}'}(0, \mathbb{Y}).$$

$$\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\underline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \underline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \underline{\mathbf{Y}'}(0, \mathbb{Y}),$$

$$\mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] = \frac{1}{r^4} \mathbb{K}[\overline{\mathbf{Y}}(\chi, \mathbb{Y})] - \frac{1}{r^3} \overline{\mathbf{Y}}(0, \mathbb{Y}) - \frac{1}{r} \overline{\mathbf{Y}'}(0, \mathbb{Y}).$$

$$\left(-\frac{1}{r} \mathbf{Y}'(0) \right) \ominus \left\{ -\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \ominus \left(-\frac{1}{r^3} \odot \mathbf{Y}(0) \right) \right\} = \left(\mathbb{K}[\underline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})], \mathbb{K}[\overline{\mathbf{Y}^{(2)}}(\chi, \mathbb{Y})] \right) = \hat{\mathbb{K}}[\mathbf{Y}^{(2)}(\chi)].$$

$$\hat{\mathbb{K}}[\mathbf{Y}^{(2)}(\chi)] = \left\{ -\frac{1}{r^3} \odot \mathbf{Y}(0) \ominus \left(-\frac{1}{r^4} \odot \hat{\mathbb{K}}[\mathbf{Y}(\chi)] \right) \right\} \ominus \frac{1}{r} \mathbf{Y}'(0).$$

(d) $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are (ii)-differentiable

$$\left(-\frac{1}{r}\Upsilon'(0)\right) \ominus \left\{\frac{1}{r^3} \odot \Upsilon(0) \ominus \frac{1}{r^4} \odot \hat{\mathbb{K}}[\Upsilon(\chi)]\right\} = \left(-\frac{1}{r}\overline{\Upsilon}'(0, \gamma) - \frac{1}{r^3}\underline{\Upsilon}(0, \gamma) + \frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)], -\frac{1}{r}\underline{\Upsilon}'(0, \gamma) - \frac{1}{r^3}\overline{\Upsilon}(0, \gamma) + \frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)]\right),$$

Since

$$\mathbb{K}[\underline{\Upsilon}^{(2)}(\chi, \gamma)] = \frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\underline{\Upsilon}(0, \gamma) - \frac{1}{r}\underline{\Upsilon}'(0, \gamma),$$

$$\mathbb{K}[\overline{\Upsilon}^{(2)}(\chi, \gamma)] = \frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\overline{\Upsilon}(0, \gamma) - \frac{1}{r}\overline{\Upsilon}'(0, \gamma).$$

Since, $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are (ii)-differentiable using Theorem (2.2):

$$\underline{\Upsilon}^{(2)}(\chi, \gamma) = \underline{\Upsilon}^{(2)}(\chi, \gamma), \overline{\Upsilon}^{(2)}(\chi, \gamma) = \overline{\Upsilon}^{(2)}(\chi, \gamma).$$

Since $\Upsilon(\chi)$ is (ii)-differentiable using Theorem (2.1) getting,

$$\overline{\Upsilon}'(0, \gamma) = \underline{\Upsilon}'(0, \gamma) \text{ and } \underline{\Upsilon}'(0, \gamma) = \overline{\Upsilon}'(0, \gamma).$$

$$\mathbb{K}[\underline{\Upsilon}^{(2)}(\chi, \gamma)] = \frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\underline{\Upsilon}(0, \gamma) - \frac{1}{r}\overline{\Upsilon}'(0, \gamma),$$

$$\mathbb{K}[\overline{\Upsilon}^{(2)}(\chi, \gamma)] = \frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\overline{\Upsilon}(0, \gamma) - \frac{1}{r}\underline{\Upsilon}'(0, \gamma).$$

$$\left(-\frac{1}{r}\Upsilon'(0)\right) \ominus \left\{\frac{1}{r^3} \odot \Upsilon(0) \ominus \frac{1}{r^4} \odot \hat{\mathbb{K}}[\Upsilon(\chi)]\right\} = \left(\mathbb{K}[\underline{\Upsilon}^{(2)}(\chi, \gamma)], \mathbb{K}[\overline{\Upsilon}^{(2)}(\chi, \gamma)]\right) = \hat{\mathbb{K}}[\Upsilon^{(2)}(\chi)].$$

$$\hat{\mathbb{K}}[\Upsilon^{(2)}(\chi)] = \left(-\frac{1}{r}\Upsilon'(0)\right) \ominus \left\{\frac{1}{r^3} \odot \Upsilon(0) \ominus \frac{1}{r^4} \odot \hat{\mathbb{K}}[\Upsilon(\chi)]\right\}.$$

5.1 Example: Consider fuzzy initial value problem:

$$\Upsilon^{(2)}(\chi) = \beta, \Upsilon(0, \gamma) = \Upsilon'(0, \gamma) = (\gamma - 1, 1 - \gamma), \beta = (\gamma - 1, 1 - \gamma), 0 \leq \gamma \leq 1.$$

Solution

Apply fuzzy Kashuri-Fundo transform on both sides, to get $\hat{\mathbb{K}}[\Upsilon^{(2)}(\chi)] = \hat{\mathbb{K}}[\beta]$.

(1) If $\Upsilon(\chi)$ and $\Upsilon'(\chi)$ are (i)-differentiable

$$\hat{\mathbb{K}}[\Upsilon^{(2)}(\chi)] = \left\{\frac{1}{r^4} \odot \hat{\mathbb{K}}[\Upsilon(\chi)] \ominus \frac{1}{r^3} \odot \Upsilon(0)\right\} \ominus \frac{1}{r}\Upsilon'(0)$$

$$\left\{\frac{1}{r^4} \odot \hat{\mathbb{K}}[\Upsilon(\chi)] \ominus \frac{1}{r^3} \odot \Upsilon(0)\right\} \ominus \frac{1}{r}\Upsilon'(0) = \hat{\mathbb{K}}[\beta]$$

$$\frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\underline{\Upsilon}(0, \gamma) - \frac{1}{r}\underline{\Upsilon}'(0, \gamma) = \mathbb{K}[\underline{\beta}],$$

$$\frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}\overline{\Upsilon}(0, \gamma) - \frac{1}{r}\overline{\Upsilon}'(0, \gamma) = \mathbb{K}[\overline{\beta}].$$

$$\frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}(\gamma - 1) - \frac{1}{r}(\gamma - 1) = r,$$

$$\frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] - \frac{1}{r^3}(1 - \gamma) - \frac{1}{r}(1 - \gamma) = r.$$

$$\frac{1}{r^4}\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] = (r + \frac{1}{r} + \frac{1}{r^3})(\gamma - 1),$$

$$\frac{1}{r^4}\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] = (r + \frac{1}{r} + \frac{1}{r^3})(1 - \gamma).$$

$$\mathbb{K}[\underline{\Upsilon}(\chi, \gamma)] = (r + r^3 + r^5)(\gamma - 1),$$

$$\mathbb{K}[\overline{\Upsilon}(\chi, \gamma)] = (r + r^3 + r^5)(1 - \gamma).$$

Using inverse Kashuri-Fundo transform,

$$\underline{\Upsilon}(\chi, \gamma) = (\gamma - 1)(1 + \chi + \frac{1}{2}\chi^2), \overline{\Upsilon}(\chi, \gamma) = (1 - \gamma)(1 + \chi + \frac{1}{2}\chi^2).$$

(2) If $\Upsilon(\chi)$ is (i)-differentiable but $\Upsilon'(t)$ is (ii)-differentiable,

$$\bar{Y}(\chi, Y) = (Y - 1) \left(\frac{1}{2} \chi^2 \right) + (1 - Y)(\chi + 1) = (1 - Y) \left(\chi + 1 - \frac{1}{2} \chi^2 \right).$$

$$\underline{Y}(\chi, Y) = (1 - Y) \left(\frac{1}{2} \chi^2 \right) + (Y - 1)(\chi + 1) = (Y - 1) \left(\chi + 1 - \frac{1}{2} \chi^2 \right).$$

(3) If $Y(\chi)$ is (ii)-differentiable and $Y'(\chi)$ is (i)-differentiable

$$\bar{Y}(\chi, Y) = (Y - 1) \left(\frac{1}{2} \chi^2 + \chi \right) + (1 - Y) = (1 - Y) \left(-\frac{1}{2} \chi^2 - \chi + 1 \right).$$

$$\underline{Y}(\chi, Y) = (1 - Y) \left(\frac{1}{2} \chi^2 + \chi \right) + (Y - 1) = (Y - 1) \left(-\frac{1}{2} \chi^2 - \chi + 1 \right).$$

(4) If $Y(\chi)$ is (ii)-differentiable and $Y'(\chi)$ is (ii)-differentiable

$$\underline{Y}(\chi, Y) = (Y - 1) \left(\frac{1}{2} \chi^2 + 1 \right) + (1 - Y)(\chi) = (Y - 1) \left(\frac{1}{2} \chi^2 + 1 - \chi \right).$$

$$\bar{Y}(\chi, Y) = (1 - Y) \left(\frac{1}{2} \chi^2 + 1 \right) + (Y - 1)(\chi) = (1 - Y) \left(\frac{1}{2} \chi^2 + 1 - \chi \right).$$

6. Conclusions

The relationship between a fuzzy function's fuzzy Kashuri-Fundo transformations and its second derivative is developed and demonstrated in this paper. The paper's primary goal is to solve fuzzy first-order differential equations and fuzzy second-order differential equations under generalized differentiability by applying the fuzzy Kashuri-Fundo transform method. Two numerical examples are provided to demonstrate the effectiveness of the suggested approach.

7. References

1. Kaleva O. The Cauchy problem for fuzzy differential equations. *Fuzzy Sets Syst.* 1990;35(3):389-396.
2. Kaleva O. Fuzzy differential equations. *Fuzzy Sets Syst.* 1987;24(3):301-317.
3. Nieto JJ, Rodriguez-Lopez R, Franco D. Linear first-order fuzzy differential equations. *Int J Uncertain Fuzziness Knowl-Based Syst.* 2006;14(06):687-709.
4. O'Regan D, Lakshmikantham V, Nieto JJ. Initial and boundary value problems for fuzzy differential equations. *Nonlinear Anal Theory Methods Appl.* 2003;54(3):405-415.
5. Abbasbandy S, *et al.* Numerical methods for fuzzy differential inclusions. *Comput Math Appl.* 2004;48(10-11):1633-1641.
6. Allahviranloo T, Ahmady N, Ahmady E. Numerical solution of fuzzy differential equations by predictor-corrector method. *Inf Sci.* 2007;177(7):1633-1647.
7. Allahviranloo T, Kiani NA, Motamedi N. Solving fuzzy differential equations by differential transformation method. *Inf Sci.* 2009;179(7):956-966.
8. Allahviranloo T, *et al.* A new method for solving fuzzy linear differential equations. *Comput.* 2011;92:181-197.
9. Alshibley STA. Solving a circuit system using fuzzy Aboodh transform. *Turk J Comput Math Educ.* 2021;12(12):3317-3323.
10. Sinha KB, Goswami D. Quantum stochastic processes and noncommutative geometry. Vol. 169. Cambridge University Press, 2007.
11. Friedman M, Ma M, Kandel A. Numerical solutions of fuzzy differential and integral equations. *Fuzzy Sets Syst.* 1999;106(1):35-48.
12. Wu HC. The fuzzy Riemann integral and its numerical integration. *Fuzzy Set Syst.* 2000;110:1-25.
13. Wu HC. The improper fuzzy Riemann integral and its numerical integration. *Inf Sci.* 1998;111(1-4):109-137.
14. ElJaoui E, Melliani S, Chadli LS. Solving a system of two fuzzy linear differential equations by fuzzy Laplace transform method. *Glob J Sci Frontier Res.* 2015;15(4):81-98.
15. Cano CY, Flores RH. On new solutions of fuzzy differential equations. *Chaos Solitons Fractals.* 2008;38(1):112-119.
16. Klir G, Yuan B. Fuzzy sets and fuzzy logic. Vol. 4. New Jersey: Prentice Hall, 1995.
17. Salahshour S, Allahviranloo T. Applications of fuzzy Laplace transforms. *Soft Comput.* 2013;17:145-158.
18. Kashuri A, Fundo A. A new integral transform. *Adv Theor Appl Math.* 2013;8(1):27-43.