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Fixed points in partially ordered metric space using δ –distance hybrid contraction

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Abstract

The main objective of this paper is to obtain some coincidence point and common fixed point results for multivalued and single valued mappings with generalized hybrid contractive conditions by employing the notion of pre-order relations, δ -distance and altering distance functions in partial ordered metric space. An example is illustrated to prove that the generalizations are real improvements.

Keywords: Hybrid contraction, multivalued maps, δ -distance, weakly compatible maps, 2010 Mathematics Subject Classification. 54H25, 47H10

Introduction

In pure and applied mathematics, metric spaces are very crucial, as metric fixed point theory has wide applications in almost all fields of quantitative sciences. The simplicity and usefulness of Banach contraction Principle ^[1] made the fixed point theorem, a very popular tool in establishing the existence of solution for an integral equation in many branches of mathematical analysis. In 1969, Nadler ^[11] extended the Banach contraction principle to multivalued mappings in complete metric spaces and one of the interesting results on weakened contractive condition was reported by Alber *et al.* ^[2]. Later, Ran and Reurings ^[15] weakened the contraction principle by considering single valued mappings over partially ordered complete metric spaces, and applied their results to matrix equations. Since then, many authors have studied the existence of fixed points in partially ordered sets. (refer ^[3, 10, 13, 14]).

Hybrid contractive conditions, that is, contractive conditions involving single valued and multivalued maps are the further addition to metric fixed point theory and its applications. For a comprehensive survey on fundamental development of hybrid contractions and historical remarks, refer S. L. Singh *et al.* ^[12, 16, 17].

The purpose of this paper is to establish some coincidence point and common fixed point results for multivalued and single valued mappings with weakened hybrid contractive conditions using the concept of pre-order relations, δ distance and altering distance functions in metric space endowed with partial order. An example is illustrated to prove that the generalizations are real improvements. These Results extend the results of Gregorio *et al.* ^[7], Giniswamy *et al.* ^[6] and many recent results in the literature.

First, some basic definitions and important results which are needed in the main results will be recalled.

Definition 1.1: A partially ordered set is a system $(X, \leq)$ where X is a non-empty set and $\leq$ is a binary relation on X satisfying, for all $x, y, z \in X$

1. $x \leq x$ (Reflexivity)
2. If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry)
3. If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitivity)

Definition 1.2. The triple $(X, d, \leq)$ is called a partially ordered metric space, if $(X, \leq)$ is a partially ordered set together with (X, d) is a metric space.

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Throughout this paper, $CB(X)$ denotes the set of all nonempty, closed and bounded subsets of a metric space X endowed with a partial order.

Definition 1.3: Let (X, d) be a metric space. The distance between any point a of X and any nonempty subset B of X is defined as

$$d(a, B) = \inf_{b \in B} d(a, b)$$

The distance between any two subsets A and B of X is given by

$$D(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\}$$

Here D is called the Hausdorff distance induced by d .

The δ -distance between any two subsets A and B of X is given by

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

By the above definition one can conclude the following:

1. $d(a, B) \leq D(A, B) \leq \delta(A, B)$
2. $d(A, b) \leq D(A, B) \leq \delta(A, B)$
3. $\delta(A, B) = 0$ if and only if $A = B = \{a\}$
4. $D(A, B) = 0$ if and only if $A = B$

Definition 1.4: Let $(X, d, \leq)$ be a partially ordered metric space, $g: X \rightarrow X$ and $T: X \rightarrow CB(X)$. If $w = gx \in Tx$ for some $x \in X$, then x is said to be a coincidence point of g and T and w is the point of coincidence of g and T .

Definition 1.5: Let $(X, d, \leq)$ be a partially ordered metric space, $g: X \rightarrow X$ and $T: X \rightarrow CB(X)$. Then a point $x \in X$ is said to be a fixed point of g and T if $x = gx \in Tx$.

The first three pre-order relations between nonempty subsets of partially ordered metric space appeared in [4, 5, 18] respectively and the last three are defined in [7].

Definition 1.6: Let $(X, d, \leq)$ be a partially ordered metric space. Let $A, B \in CB(X)$, then the pre-order relations are defined as follows:

1. $A <_1 B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \leq b$,
2. $A <_2 B \Leftrightarrow \forall b \in B, \exists a \in A$ such that $a \leq b$,
3. $A <_3 B \Leftrightarrow A <_1 B$ and $A <_2 B$,
4. $A <_4 B \Leftrightarrow \forall a \in A, \exists b \in B$ such that $a \leq b$ and $d(a, b) \leq D(A, B)$,
5. $A <_5 B \Leftrightarrow \forall b \in B, \exists a \in A$ such that $a \leq b$ and $d(a, b) \leq D(A, B)$,
6. $A <_6 B \Leftrightarrow A <_4 B$ and $A <_5 B$.

Definition 1.7: Let $(X, d, \leq)$ be a partially ordered metric space, $g: X \rightarrow X$ and $T: X \rightarrow CB(X)$. Then the mappings g and T are called weakly compatible if $gx \in Tx$ for some $x \in X$ implies $gT(x) \subseteq Tg(x)$.

The altering distance function is defined as follows and it has been utilized in many works in metric fixed point theory.

Definition 1.8: [9] A function $\psi: [0, \infty) \rightarrow [0, 1)$ is called an altering distance function if the following properties are satisfied:

1. ψ is monotone increasing and continuous,
2. $\psi(t) = 0$ if and only if $t = 0$.

Lemma 1.9: [11]. Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n \rightarrow \infty} D(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ then $x \in A$.

Proposition 1.10: [8]. Let X be a given nonempty set. Assume that $g: X \rightarrow X$ and $T: X \rightarrow 2^X$ are weakly compatible mappings. If g and T have a unique point of coincidence $w = gx \in Tx$, then w is the unique common fixed point of g and T .

Main Results

We now state and prove the first main result. Throughout this paper, Φ denotes the class of functions $\phi: [0, \infty) \rightarrow [0, 1)$ such that $\lim_{r \rightarrow t^+} \sup \phi(r) < 1$ for each $t \in [0, \infty)$

Theorem 2.1: Let $(X, d, \leq)$ be partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping and $g: X \rightarrow X$ be a self map satisfying the following conditions

1. $T(x) \subseteq gX \forall x \in X$ and gX is complete

2. There exists $x_0 \in X$ such that $\{gx_0\} <_1 Tx_0$,
3. If $gx \leq gy$ then $Tx <_4 Ty$ for all $x \leq y$ in X ,
4. $\psi(\delta(Tx, Ty)) \leq \phi(d(gx, gy)) \psi(d(gx, gy)) \forall x, y \in X$ where $\phi \in \Phi$.

Then T and g have a unique coincidence point. Further, if T and g are weakly compatible, then T and g have a unique common fixed point.

Proof: Let x_0 be any point in X such that $\{gx_0\} <_1 Tx_0$.

Then there exists $u \in Tx_0$ such that $gx_0 \leq u$. Since $Tx_0 \subseteq gX$, $u \in gX$. That is, there exists a point $x_1 \in X$ such that $u = gx_1$. Therefore $gx_0 \leq gx_1$ which implies $Tx_0 <_4 Tx_1$.

Since $gx_1 \in Tx_0$ there exists $v \in Tx_1$ such that $gx_1 \leq v$ and $d(gx_1, v) \leq D(Tx_0, Tx_1)$.

Again, since $Tx_1 \subseteq gX$, $v \in gX$. i.e., there exists a point $x_2 \in X$ such that $v = gx_2$.

Therefore $gx_1 \leq gx_2$ and

$$d(gx_1, gx_2) \leq D(Tx_0, Tx_1) \leq \delta(Tx_0, Tx_1)$$

From the monotone property of ψ , we get

$$d(gx_1, gx_2) \leq d(gx_0, gx_1).$$

Now $gx_1 \leq gx_2$ implies $Tx_1 <_4 Tx_2$. Using the same argument as above we get,

$$gx_2 \leq gx_3 \text{ and } d(gx_2, gx_3) \leq d(gx_1, gx_2).$$

Continuing this process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} \in Tx_n$ and

$$d(gx_n, gx_{n+1}) \leq d(gx_{n-1}, gx_n)$$

Therefore $\{d(gx_n, gx_{n+1})\}$ is a decreasing sequence and bounded below by 0 and hence it is convergent. Let $\lim_{n \rightarrow \infty} d(gx_n, gx_{n+1}) = r$. Clearly $r = 0$.

Thus $d(gx_n, gx_m) \rightarrow 0$ as $n \rightarrow \infty$.

Therefore, $\{gx_n\}$ is a Cauchy sequence. Since gX is complete, there exists a point $gp \in gX$ such that $gx_n \rightarrow gp$. Then $D(Tx_n, Tp) \leq \delta(Tx_n, Tp)$

From the monotone property of ψ , we get

$$D(Tx_n, Tp) \leq \delta(Tx_n, Tp) \leq d(gx_n, gp) \text{ which gives } \lim_{n \rightarrow \infty} D(Tx_n, Tp) = 0.$$

Hence by Lemma 1.7 we get $gp \in Tp$. Thus p is the coincidence point of T and g .

To prove the uniqueness: Let t and s be the two points of coincidence of T and g . Then, $t = gp \in Tp$ and $s = gq \in Tq$.

$$\text{Consider } \psi(\delta(Tp, Tq)) \leq \phi(d(gp, gq)) \psi(d(gp, gq)) \leq \psi(d(gp, gq)) \leq \psi(\delta(Tp, Tq))$$

$$\delta(Tp, Tq) \leq \delta(Tp, Tq)$$

Which implies $Tp = Tq = \{t\} = \{s\}$

Therefore, t is the unique point of coincidence of T and g .

Since T and g are weakly compatible, by Proposition 1.10 they have a unique common fixed point.

Example 2.2: Let $X = [0, 20)$ and d be the usual metric on X . Let $\leq$ be the partial order on X . Then $(X, d, \leq)$ is a partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping defined by

$$T(x) = \begin{cases} \left\{ \frac{1}{2}x^2 \right\} & \text{if } 0 \leq x < 1 \\ \left\{ \frac{x}{x+1} \right\} & \text{if } x \geq 1 \end{cases}$$

$$\text{Let } g: X \rightarrow X \text{ be defined by } g(x) = \begin{cases} 10x & \text{if } 0 \leq x < 1 \\ \frac{15}{x} & \text{if } x \geq 1 \end{cases}$$

Then $T(x) \subseteq gX \forall x \in X$ and gX is complete. Here T and g satisfies the condition (4) of Theorem 2.1 for $\psi(t) = \frac{1}{2}t$ and for any $\phi \in \Phi$.

Note that $x = 0$ is the unique coincidence point of T and g .

Further, T and g are weakly compatible and satisfy all the conditions of Theorem 2.1 and have a unique common fixed point $x = 0$.

The above example shows the applicability of the above theorem, where Theorem 3.1 of [7] fails. That is, Theorem 3.1 of [7] cannot be applied to the above example since X is not complete.

A dual result of Theorem 2.1 can be obtained by replacing $<_1$ and $<_4$ relations by $<_2$ and $<_5$ respectively.

Theorem 2.3: Let $(X, d, \leq)$ be partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping and $g: X \rightarrow X$ be a self map satisfying the following conditions

1. $T(x) \subseteq gX \forall x \in X$ and gX is complete
2. There exists $x_0 \in X$ such that $Tx_0 <_2 \{gx_0\}$
3. If $gx \leq gy$ then $Tx <_5 Ty$ for all $x \leq y$ in X ,
4. $\psi(\delta(Tx, Ty)) \leq \phi(d(gx, gy)) \psi(d(gx, gy)) \forall x, y \in X$ where $\phi \in \Phi$.

Then T and g have a unique coincidence point. Further, if T and g are weakly compatible, then T and g have a unique common fixed point.

Proof: Let x_0 be any point in X such that $Tx_0 <_2 \{gx_0\}$

Then there exists $u \in Tx_0$ such that $u \leq gx_0$. Since $Tx_0 \subseteq gX$, $u \in gX$. That is, there exists a point $x_1 \in X$ such that $u = gx_1$.

Therefore $gx_1 \leq gx_0$ implies $Tx_1 <_5 Tx_0$.

Since $gx_1 \in Tx_0$ there exists $v \in Tx_1$ such that $v \leq gx_1$.

Again, since $Tx_1 \subseteq gX$, $v \in gX$. i.e., there exists a point $x_2 \in X$ such that $v = gx_2$.

Therefore $gx_2 \leq gx_1$ and

$$d(gx_2, gx_1) \leq D(Tx_1, Tx_0) \leq \delta(Tx_1, Tx_0)$$

From the monotone property of ψ , we get

$$d(gx_2, gx_1) \leq d(gx_1, gx_0).$$

Now $gx_2 \leq gx_1$ implies $Tx_2 <_5 Tx_1$. Using the same argument as above we get,

$$gx_3 \leq gx_2 \text{ and } d(gx_3, gx_2) \leq d(gx_2, gx_1).$$

Continuing this process, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} \in Tx_n$ and

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1})$$

Therefore $\{d(gx_{n+1}, gx_n)\}$ is a decreasing sequence and bounded below by 0 and hence it is convergent. Let

$$\lim_{n \rightarrow \infty} d(gx_{n+1}, gx_n) = r. \text{ Clearly } r = 0.$$

Thus $d(gx_n, gx_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

Therefore, $\{gx_n\}$ is a Cauchy sequence. Since gX is complete, there exists a point $gp \in gX$ such that $gx_n \rightarrow gp$. Then

$$D(Tx_n, Tp) \leq \delta(Tx_n, Tp)$$

From the monotone property of ψ , we get

$$D(Tx_n, Tp) \leq \delta(Tx_n, Tp) \leq d(gx_n, gp) \text{ which gives } \lim_{n \rightarrow \infty} D(Tx_n, Tp) = 0.$$

Hence by Lemma 1.7 we get $gp \in Tp$. Thus p is the coincidence point of T and g .

To prove the uniqueness: Let t and s be the two points of coincidence of T and g . Then,

$$t = gp \in Tp \text{ and } s = gq \in Tq.$$

$$\text{Consider } \psi(\delta(Tp, Tq)) \leq \phi(d(gp, gq)) \psi(d(gp, gq)) \leq \psi(d(gp, gq)) \leq \psi(\delta(Tp, Tq))$$

$$\delta(Tp, Tq) \leq \delta(Tp, Tq)$$

Which implies $Tp = Tq = \{t\} = \{s\}$

Therefore, t is the unique point of coincidence of T and g .

Since T and g are weakly compatible, by Proposition 1.10 they have a unique common fixed point.

Example 2.4: Let $X = [0, \infty)$ and d be the usual metric on X . Let $\leq$ be the partial order on X . Then $(X, d, \leq)$ is a partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping defined by.

$$T(x) = \begin{cases} \left\{ \frac{1}{3}x^2 \right\} & \text{if } 0 \leq x \leq 1 \\ \left\{ \frac{x}{x+1} \right\} & \text{if } x > 1 \end{cases}$$

Let $g: X \rightarrow X$ be defined by $g(x) = 10x \forall x$

Then $T(x) \subseteq gX \forall x \in X$ and gX is complete. Here T and g satisfies the condition (4) of Theorem 2.3 for $\psi(t) = 5t$ and for any $\phi \in \Phi$.

Note that $x = 0$ is the unique coincidence point of T and g .

Further, T and g are weakly compatible and satisfy all the conditions of Theorem 2.3 and have a unique common fixed point $x = 0$.

Remark 2.5. Note that Theorem 2.1 generalizes Theorem 3.1 of [7] and Theorem 2.1 of [6], while Theorem 2.3 generalizes Theorem 3.3 of [7] and Theorem 2.3 of [6]. Taking g as the identity map in the above theorems, we get the following corollaries which are sharpened versions of results of [7] and [6].

Corollary 2.6: Let $(X, d, \leq)$ be partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping satisfying the following conditions

1. There exists $x_0 \in X$ such that $\{x_0\} <_1 Tx_0$,
2. If $x \leq y$ then $Tx <_4 Ty$ for all $x \leq y$ in X ,
3. $\psi(\delta(Tx, Ty)) \leq \phi(d(x, y)) \psi(d(x, y)) \forall x, y \in X$ where $\phi \in \Phi$.

Then T has a unique fixed point.

Corollary 2.7: Let $(X, d, \leq)$ be partially ordered metric space. Let $T: X \rightarrow CB(X)$ be a multivalued mapping satisfying the following conditions

1. There exists $x_0 \in X$ such that $Tx_0 <_2 \{x_0\}$
2. If $x \leq y$ then $Tx <_5 Ty$ for all $x \leq y$ in X ,
3. $\psi(\delta(Tx, Ty)) \leq \phi(d(x, y)) \psi(d(x, y)) \forall x, y \in X$ where $\phi \in \Phi$.

Then T has a unique fixed point.

As a consequence of Theorem 2.1 and Theorem 2.3 we have the following fixed point theorem for single valued mappings which can be proved easily.

Theorem 2.8: Let $(X, d, \leq)$ be partially ordered metric space. Let $T, g: X \rightarrow X$ be two self maps satisfying the following conditions

1. $T(x) \subseteq gX \forall x \in X$ and gX is complete
2. There exists $x_0 \in X$ such that $Tx_0 \leq gx_0$ or $gx_0 \leq Tx_0$
3. If $gx \leq gy$ then $Tx \leq Ty$ for all $x \leq y$ in X ,
4. $\psi(d(Tx, Ty)) \leq \phi(d(gx, gy)) \psi(d(gx, gy)) \forall x, y \in X$ where $\phi \in \Phi$.

Then T and g have a unique coincidence point. Further, if T and g are weakly compatible, then T and g have a unique common fixed point.

References

1. Banach S. Sur les operations dans les ensembles abstraites et leurs applications. Fund. Math. 1922;3:133-181.
2. Alber Ya I, Guerre-Delabriere S. Principles of weakly contractive maps in Hilbert space. New Results on Operation Theory: Adv. Appl. 1997;98:7-22.
3. Beg I, Butt AR. Fixed point theorems for set valued mappings in partially ordered metric spaces. Int. J. Math. Sci. 2003;7:66-68.
4. Beg I, Butt AR. Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces. Math. Commun. 2012;15:65-76.
5. Choudhury BS, Metiya N. Multivalued and single valued fixed point results in partially ordered metric spaces. Arab J. Math. Sci. 2011;17:135-151.
6. Giniswamy P, Maheshwari C, Jeyanthi. Fixed points on hybrid contractive conditions in partially ordered metric space. Journal of Nonlinear and Convex Analysis. 2016;17(6):1137-1149.
7. Gregorio RO, Macansantos PS. Fixed point theorems and stability of fixed point sets of multivalued mappings. Adv. Fixed Point Theory. 2013;3:735-746.
8. Kaewcharoen A, Kaewkhao A. Common fixed points for singlevalued and multivalued mappings in G-metric spaces. Int. J. Math. Anal. 2011;5:1775-1790.
9. Khan MS, Swaleh M, Sessa S. Fixed point theorems by altering distances between the points. Bull. Australian Math. Soc. 1984;30:1-9.
10. Klim D, Wardowski D. Fixed point theorems for set valued contractions in complete metric spaces. J. Math. Anal. Appl. 2007;334:132-139.
11. Nadler SB. Multivalued contraction mappings. Pacific J. Math. 1969;30:475-478.
12. Naimpally SA, Singh SL, Whitfield JHM. Coincidence point theorems for hybrid contractions. Math. Nachr. 1986;127:177-180.
13. Nieto JJ, Rodriguez-Lopez R. Contractive mapping theorems in partially ordered sets and application to ordinary differential equations. Order. 2005;22:223-239.
14. Nieto JJ, Rodriguez-Lopez R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. 2007;23:2205-2212.
15. Ran ACM, Reurings MCB. A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc. 2004;132:1435-1443.
16. Singh SL, Mishra SN. Nonlinear hybrid contractions. J. Nat. Phys. Sci. 1994;5(8):191-206.
17. Singh SL, Mishra SN. Coincidence and fixed points of non self hybrid contractions. J. Math. Anal. Appl. 2001;256:486-497.
18. Straccia U, Ojeda-Aciego M, Damasio CV. On fixed points of multivalued functions on complete lattices and their applications to generalized logic programs. SIAM J. Comput. 2009;38:1881-1911.