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Some results on n-power operators

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Abstract

It is a well known result that 1-power normal operator is equivalent to normal operators. This paper will discuss some results on n-power normal operators on the Hilbert space. It is shown that if T is a diagonal matrix, then T is n-power normal for any integer $n > 2$. We show that the product of two commuting 2-power normal operators is 2-power normal if each operator commutes with the adjoint of the other operator.

Keywords: n-power normal operators, partial isometry, diagonal matrix

Introduction

In this section, we define terms, terminologies and the notations used.

We denote Hilbert space with H and $B(H)$ to mean a Banach algebra of all bounded linear operators. Suppose $T \in B(H)$, then T^* is the adjoint of T . An operator T is a self-adjoint operator if $T = T^*$. $T \in B(H)$ is normal if $TT^* = T^*T$ and n-power normal if $T^n T^* = T^* T^n$.

Remark

1- power normal operators are normal operator

Definition

An operator T is 2-power normal if $T^2 T^* = T^* T^2$ which are denoted by [2N].

Theorem

An operator $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{C}$ is said to be 2-power normal if and only if $(a + d) = 0$ and $|b| = |c|$ or $b(d - a) = c(d - a)$.

Proof

$$\begin{aligned} T^2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{pmatrix} \\ T^* &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ T^2 T^* &= \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} a^3 + abc + ab^2 + bd^2 & a^2 c + bc^2 + abd + bd^2 \\ a^2 c + acd + b^2 c + bd^2 & ac^2 + dc^2 + b^2 c + d^3 \end{pmatrix} \\ T^* T^2 &= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{pmatrix} \end{aligned}$$

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$$= \begin{pmatrix} a^3 + abc + ac^2 + dc^2 & a^2b + abd + bc^2 + cd^2 \\ ba^2 + b^2c + acd + d^2c & ab^2 + b^2d + bcd + d^3 \end{pmatrix}$$

Then $T^2T^* = T^*T^2$ implies that

$$a^3 + abc + ab^2 + b^2d = a^3 + abc + ac^2 + dc^2$$

$$\Rightarrow ab^2 + b^2d = ac^2 + dc^2$$

$$= b^2(a + d) = c^2(a + d)$$

$$\Rightarrow b^2(a + d) - c^2(a + d) = 0$$

$$= (b^2 - c^2)(a + d) = 0$$

$$b^2 - c^2 = 0 \text{ or } a + d = 0$$

$$\Rightarrow |b| = |c| \text{ or } a + d = 0$$

Example

$$\text{Let } T = \begin{pmatrix} 1+i & 2+i \\ 3+2i & 4+3i \end{pmatrix}$$

$$\text{Then } T^2 = T \cdot T = \begin{pmatrix} 1+i & 2+i \\ 3+2i & 4+3i \end{pmatrix} \cdot \begin{pmatrix} 1+i & 2+i \\ 3+2i & 4+3i \end{pmatrix}$$

$$T^2 = \begin{pmatrix} (1+i)^2 + (2+i)(3+2i) & (1+i)(2+i) + (2+i)(4+3i) \\ (3+2i)(1+i) + (4+3i)(3+2i) & (3+2i)(2+i) + (4+3i)^2 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 1+9i & -1+9i \\ -5+21i & -1+17i \end{pmatrix}$$

$$\text{then } T^* = \begin{pmatrix} 1-i & 3-2i \\ 2-i & 4-3i \end{pmatrix}$$

$$T^* \cdot T^2 = \begin{pmatrix} 1-i & 3-2i \\ 2-i & 4-3i \end{pmatrix} \begin{pmatrix} 1+9i & -1+9i \\ -5+21i & -1+17i \end{pmatrix}$$

$$= \begin{pmatrix} 16+8i & 0-18i \\ 0-18i & 0+0i \end{pmatrix}$$

$$T^2 \cdot T^* = \begin{pmatrix} 1+9i & -1+9i \\ -5+21i & -1+17i \end{pmatrix} \begin{pmatrix} 1-i & 3-2i \\ 2-i & 4-3i \end{pmatrix}$$

$$T^2 \cdot T^* = \begin{pmatrix} 16+8i & 0-18i \\ 0-18i & 0+0i \end{pmatrix}$$

We have verified that $T^2T^* = T^*T^2$ for the matrix T given above. Therefore, T is a 2-power normal operator.

Theorem

Let T be a diagonal matrix. Then T is n-power normal for all integers $n \geq 2$

Proof

$$\text{Let } T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \text{ where } a, b \in \mathbb{C}$$

$$\text{Then, } T^2 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \text{ and } T^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\text{This implies that } T^2T^* = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$$

$$\text{Also, } T^*T^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

$$= \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$$

Hence T is 2-power normal.

Now suppose that T is 3-power normal, then

$$T^3 = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix} \text{ and } T^* = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\begin{aligned} \text{Then } T^3 T^* &= \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \\ &= \begin{pmatrix} a^4 & 0 \\ 0 & b^4 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{Also, } T^* T^3 &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix} \\ &= \begin{pmatrix} a^4 & 0 \\ 0 & b^4 \end{pmatrix} \end{aligned}$$

Hence T is n-power normal for all $n \geq 2$

Theorem

The product of two commuting normal operators T and S is normal if T commutes with S^* and S commutes with T^* .

Proof

Consider $(TS)(TS)^* = (TS)(T^*S^*)$

$$= TSS^*T^*$$

$$= TS^*ST^*$$

$$= S^*TT^*S$$

$$= S^*T^*TS$$

$$= (TS)^*(TS)$$

This implies that TS is a normal operator

We can extend this idea to 2-power normal operators

Theorem

The product of two commuting 2-power normal operators T and S is 2-power normal if T commutes with S^* or S commutes with T^*

Proof

Note, $(ST)^2(ST)^* = (ST)^2(T^*S^*)$

$$= S^2(T^2T^*)S^*$$

$$= S^2(T^*T^2)S^*$$

$$= (S^2T^*)(T^2S^*)$$

$$= (T^*S^2)(T^2S^*)$$

$$= T^*(S^2T^2)S^*$$

$$= (T^*T^2)(S^2S^*)$$

$$= (T^*T^2)(S^*S^2)$$

$$= T^*(T^2 S^*) S^2$$

$$= (T^* S^*) (T^2 S^2)$$

$$= (ST)^* (ST)^2$$

Example

Lets illustrate the theorem with a numerical example using 2x2 complex matrix

$$\text{Let } T = \begin{pmatrix} 1+i & 2 \\ 0 & 1-i \end{pmatrix} \text{ and } S = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$$

First, we verify that T and S are 2-power normal

$$T^* = \begin{pmatrix} 1-i & 0 \\ 2 & 1+i \end{pmatrix}$$

$$T^2 = \begin{pmatrix} 1+i & 2 \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} 1+i & 2 \\ 0 & 1-i \end{pmatrix}$$

$$= \begin{pmatrix} 2i & 4+2i \\ 0 & -2i \end{pmatrix}$$

$$T^2 T^* = \begin{pmatrix} 2i & 4+2i \\ 0 & -2i \end{pmatrix} \begin{pmatrix} 1-i & 0 \\ 2 & 1+i \end{pmatrix}$$

$$= \begin{pmatrix} 2i & 4+2i \\ 0 & -2i \end{pmatrix}$$

$$T^* T^2 = \begin{pmatrix} 1-i & 0 \\ 2 & 1+i \end{pmatrix} \begin{pmatrix} 2i & 4+2i \\ 0 & -2i \end{pmatrix}$$

$$= \begin{pmatrix} 2i & 4+2i \\ 0 & -2i \end{pmatrix}$$

Since $T^2 T^* = T^* T^2$, T then T is 2-power normal.

$$S^* = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} \text{ and } S^2 = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2i+1 \\ 2i+1 & 2 \end{pmatrix}$$

$$S^2 S^* = \begin{pmatrix} 5 & -2i+1 \\ 2i+1 & 2 \end{pmatrix} \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -4i \\ 2i+2 & 3 \end{pmatrix}$$

$$S^* S^2 = \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 5 & -2i+1 \\ 2i+1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & -4i \\ 2i+2 & 3 \end{pmatrix}$$

Since $S^2 S^* = S^* S^2$, S is 2-power normal.

$$TS = \begin{pmatrix} 2+2i & 2-i \\ 0 & 1-i \end{pmatrix} \text{ and } ST = \begin{pmatrix} 2+2i & 2-i \\ 0 & 1-i \end{pmatrix}$$

Since $TS = ST$, T and S commute.

We verify that TS is 2-power normal.

$$(TS)^2 = \begin{pmatrix} 2+2i & 2-i \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} 2+2i & 2-i \\ 0 & 1-i \end{pmatrix}$$

$$= \begin{pmatrix} 4+8i & 4-2i \\ 0 & 1-2i \end{pmatrix}$$

$$(TS)^* = \begin{pmatrix} 2-2i & 2+i \\ 0 & 1+i \end{pmatrix}$$

$$(TS)^2(TS)^* = \begin{pmatrix} 4+8i & 4-2i \\ 0 & 1-2i \end{pmatrix} \begin{pmatrix} 2-2i & 2+i \\ 0 & 1+i \end{pmatrix}$$

$$= \begin{pmatrix} 2+8i & 4-2i \\ 0 & 1-2i \end{pmatrix}$$

$$(TS)^*(TS)^2 = \begin{pmatrix} 2-2i & 2+i \\ 0 & 1+i \end{pmatrix} \begin{pmatrix} 4+8i & 4-2i \\ 0 & 1-2i \end{pmatrix}$$

$$= \begin{pmatrix} 2+8i & 4-2i \\ 0 & 1-2i \end{pmatrix}$$

Since $(TS)^2(TS)^* = (TS)^*(TS)^2$, TS is 2-power normal.

Theorem

The product of two operators S, T is 2-normal if $ST + TS = 0$ and S^2 commutes with T^* and S^* commutes with T^2 .

Proof

We note that $ST = -TS$ from the above equation.

$$(ST)^2(ST)^* - (ST)^*(ST)^2$$

$$= (ST)^2(T^*S^*) - (T^*S^*)(ST)^2$$

$$= (ST)(ST)(T^*S^*) - (T^*S^*)(ST)(ST)$$

$$= S(TS)T(T^*S^*) - (T^*S^*)S(TS)T$$

$$= -S(ST)T(T^*S^*) + (T^*S^*)S(ST)T$$

$$= -(S^2T^2)(T^*S^*) + (T^*S^*)(S^2T^2)$$

$$= -S^2T^*T^2S^* + T^*S^2T^2S^*$$

$$= -T^*S^2T^2S^* + T^*S^2T^2S^* = 0$$

Hence ST is a 2-power normal

Example

$$\text{Let } S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$ST = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$TS = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$ST + TS = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, $ST + TS = 0$.

We show that S^2 commutes with T^*

$$S^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$T^* = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

Since $-I$ is a scalar multiple of the identity matrix, it commutes with any matrix, including T^* .

We now show that S^* commutes with T^2

$$S^* = S^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

Again, since $-I$ is a scalar multiple of the identity matrix, it commutes with any matrix, including S^* .

Theorem

If T is a 2-power normal operator and also partial isometry, then T is n -power normal for all integers $n \in B(H)$

Proof

Let T be 2-normal and partial isometry. This implies that

$$T^2 T^* = T^* T^2 \text{ and } T = T T^* T$$

$$\text{So } T^3 T^* = T(T^2 T^*)$$

$$= T(T^* T^2)$$

$$= T(T^* T T)$$

Since T is partial isometry, we have

$$= T^2$$

$$\text{Also } T^* T^3 = (T^* T^2) T$$

$$= (T^2 T^*) T$$

Since T is a partial isometry, we have

$$= T(T T^* T)$$

$$= T^2$$

Hence T is 3-normal

Now suppose that T^k is k -normal for all integers $k = 2, 3, \dots$

$$\text{Then } T^{k+1} T^* = T^{k-1} (T^2 T^*)$$

$$= T^{k-1} (T^* T^2)$$

$$= (T^{k-1} T^*) T^2$$

$$= (T^* T^{k-1}) T^2$$

$$= (T^* T^{k-1}) T^2$$

$$= T^* T^{k+1}$$

Hence T is $(k+1)$ -normal for all $k \geq 2$

Theorem

An operator $T \in B(H)$ is n -normal if and only if T^n is normal for some integer n .

Proof

If T is n -power normal, then this implies that $T^n T^* = T^* T^n$.

This shows that $T^n(T^*)^n = T^*T^n(T^*)^{n-1}$

$$=T^*(T^nT^*)(T^*)^{n-1}$$

$$=T^*T^nT^*(T^*)^{n-2}$$

$$=(T^*)^nT^n.$$

Hence T is n-power normal.

Conversely, let T^n be normal, then $T^nT = TT^n$. This implies that by Fuglede theorem that $(T^n)^*T = T(T^n)^*$ that is $T^*T^n = T^nT^*$. Hence T is n-normal.

Theorem

If S, T are n-power normal operators which commute, then ST is an n-power normal operator

Proof

Since S, T are commuting n-power normal operators,

S^n, T^n are commuting normal operators. So S^nT^n is a normal operator.

Since $S^nT^n = (ST)^n$, $(ST)^n$ is normal. ST is n-normal.

References

1. Jibril AAS. On nnn-power normal operators. J Sci Eng. 2008;33(2A):247-251.
2. Alzuraiqi SA, Patel AB. On normal operators. Gen Math Notes. 2010;1(2):61-73.
3. Alzuraiqi SA, Patel AB. On nnn-normal operators. Gen Math Notes. 2010;1(2):61-73.
4. Conway JB. A Course in Functional Analysis. 2nd ed. New York: Springer-Verlag; c1990.
5. Jibril AAS. On nnn-power normal operators. J Sci Eng. 2008;33(2):247-251.
6. Furuta T. A remark on a class of operators. Proc Japan Acad. 1967;43:607-609.
7. Rudin W. Functional Analysis. New York: McGraw-Hill; c1973.
8. Agler J, Stankus M. mmm-isometries transformations of Hilbert space I. Integral Equations Oper Theory. 1995;21(3):383-427.
9. Conway JB. A Course in Functional Analysis. New York: Springer; c1981.
10. Halmos PR. Normal dilations and extensions of operators. Summa Bras Math. 1952;2:55-62.
11. Brown A. On a class of operators. Proc Amer Math Soc. 1953;4(5):723-728.