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Fuzzy slicing structure property with some results

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Abstract

This paper aims to structure a new concept of fuzzy Slicing Structure Property denoted by (FSSP via short) and we give some results, also we prove the fuzzy fibration is not fuzzy Hurewics fibration.

Keywords: Fuzzy fibration, fuzzy slicing structure, fuzzy g-bundle, fuzzy lifting homotopy property, fuzzy locally equi-connected

1. Introduction

The fundamental distinction from modern conceptions of fuzzy fiber spaces. Seifert believed to be now known as the base space (fuzzy topological space) of fuzzy fiber (topological) space μ_Y was a quotient space of μ_Y and not a component of the structure. Hassler Whitney first described fuzzy fiber space ^[3] under the term sphere space. Later, he modified the name to sphere bundle ^[14]. J Mahalakshmi, M Sudha ^[9], and others are credited with the theory of fuzzy fiber spaces, fuzzy principal bundles, fuzzy vector bundles, and fuzzy fibrations as special cases. A fuzzy fiber structure (μ_X, π, μ_Y) is a triple consisting of two fuzzy topological spaces μ_X, μ_Y and $\pi: \mu_X \rightarrow \mu_Y$ a continuous surjection. The total fuzzy (or fibered) space refer to space μ_X , π is termed the projection, and μ_Y is the base space for each $y_0 \in \mu_Y$ the set $F = \pi^{-1}(y_0)$ and F is called fuzzy fiber over y_0 . The fuzzy fiber structure over μ_Y refers to (μ_X, π, μ_Y) ^[9]. A fuzzy fibration is a continuous mapping the fuzzy satisfying lifting homotopy property, and its CW-complex is called fuzzy Serre fibration. The slicing structure property (SSP) introduced by ^[4] if we have given map $p: E \rightarrow X$ has the SSP then E is called sliced fiber space over X relative to p and in particular, every bundle space is a sliced fiber space. In addition, the triple (E, p, X) is called a Hu fiber space (for more details see) ^[5]. We investigate that most of the theorems that are valid for fuzzy fibration (fuzzy Serre fibration) ^[4, 6]. In particular, with fuzzy slicing structure property.

2. Preliminaries

In this section, we illustrate some basic definitions, properties, and results.

Definition 2.1 ^[12] Let S and T be topological spaces. If p and q are mapping of S into T , then p and q are homotopic ($p \simeq q$) if there exists a mapping $h: S \times I^1 \rightarrow T$ via $h(e, 0) = p(e)$ and $h(e, 1) = q(e)$ for all $e \in S$. The mapping h is called a homotopy between p and q .

Definition 2.2 ^[8, 2] Let $p: E \rightarrow B$ be a map (fiber bundles) from topological space E to topological space B , we say that p has lifting Homotopy Property (L.H.P. by short) with respect to X iff given a map $f: X \rightarrow E$ and $h_t: X \rightarrow B$ is homotopy where $p \circ f = h_0$. Then there exist a homotopy $h_t^*: X \rightarrow E$ such that (1) $h_0^* = f$. (2) $p \circ h_t^* = h_t$ for all $x \in X$ and $t \in I$. Where I is the unit interval.

1. The map p is called Hurewics (Serre) fibration if it has lifting homotopy property with respect to all (CW-complex Y) spaces.
2. The map p is fiber space which called fibration if it has lifting homotopy property with respect to classes of space.

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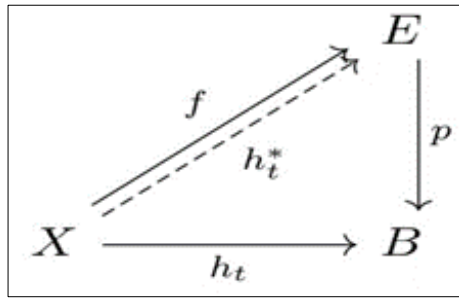


Fig 1: Lifting Homotopy Property

Definition 2.3 ^[10] A fuzzy set in set X is ordered pair $(X, \mu_A(X))$ such that $\mu_A(X)$ called membership with domain X and the co-domain is value in $I = [0,1]$, denoted by $= \{(x, \mu_A(x)) : x \in X\}$

Definition 2.4 ^[3] Let $\mu_A \in \mathfrak{S}(E)$ all the fuzzy set from topological space (E, τ) and $\tau^* \in \mathfrak{S}(E)$. Let $[\tau^* = \{G : v \rightarrow I : v \in \tau\}, \tau^* = \{G_\alpha : G \in \tau^*, \alpha \in I\}]$. The pair (μ_A, τ^*) is called fuzzy topological space, if and only if

- $\emptyset, E \in \tau^*$
- If $G, E \in \tau^*$, then $G \cap E \in \tau^*$
- If $\{G_i\}_{i \in \Lambda} \in \tau^*$, then $\cup_{i \in \Lambda} G_i \in \tau^*$

Definition 2.5 Let (μ_A, τ_1^*) , and (μ_B, τ_2^*) be fuzzy topological spaces, and $u, v: \mu_A \rightarrow \mu_B$ are fuzzy continuous functions. u is a fuzzy homotopy to v if there exist a fuzzy continuous function which is a family of maps $h_t: \mu_A \rightarrow \mu_B$ such that for all $t \in I$

$$h_t(x) = u_t(x), h_0(x) = u(x) \text{ and } h_1(x) = v(x)$$

If u and v are fuzzy homotopic functions, we write $u \sim v$.

Definition 2.6 Let (μ_E, π, μ_B) be fuzzy fiber space where fuzzy section s is a fuzzy continuous $s: \mu_B \rightarrow \mu_E$ such that $\pi \circ s = id_{\mu_B}$.

Definition 2.7 A fuzzy fiber structure is a triple (μ_X, π, μ_Y) consisting of two fuzzy topological spaces μ_X, μ_Y and a fuzzy continuous surjection $\pi: \mu_X \rightarrow \mu_Y$. The space μ_X is called fuzzy total [or fuzzy fibered] space, π is termed the projection, and μ_Y is a fuzzy base space. Then $F = \pi^{-1}(y)$ and F is called fuzzy fiber over y . We refer to (μ_X, π, μ_Y) as a fuzzy fiber structure over π .

Definition 2.8 Let $\pi: \mu_X \rightarrow \mu_Y$ be a fuzzy map. We say that π has a Fuzzy lifting Homotopy Property (F.L.H.P) with respect to μ_Y if and only if given a fuzzy map $u: \mu_Z \rightarrow \mu_X$ and a fuzzy homotopy $h_t: \mu_Z \rightarrow \mu_Y$ such that $\pi \circ u = h_0$. Then there exists a fuzzy homotopy $h_t^*: \mu_Z \rightarrow \mu_X$ such that:

- $h_0^* = u$
- $\pi \circ h_t^* = h_t$ for all $y \in \mu_Y$ and $t \in I$

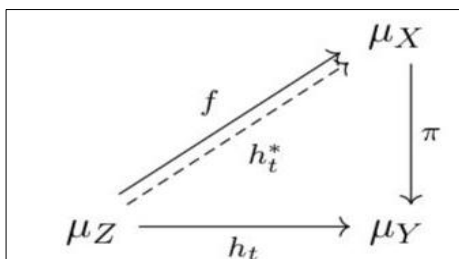


Fig 2: Fuzzy lifting Homotopy Property

- The fuzzy map π is called fuzzy Hurewics (Serre) fibration if it has fuzzy lifting homotopy property with respect to all (CW-complex μ_Y) spaces.
- The fuzzy fiber space has fuzzy lifting homotopy property with respect to classes of space which called fuzzy fibration.

3. Non-trivial of fuzzy principal bundle

In this section, we give examples for the fuzzy projection map which is fuzzy fibration but not a fuzzy Hurewics fibration. fuzzy projection $\pi: \mu_X \rightarrow \mu_Y$, where gives a fuzzy trivial G-bundle over μ_Y since $\mu_X|_U \simeq \mu_{X_U}$, and $\mu_X|_V \simeq \mu_{X_V}$ are fuzzy trivial G-bundles. To prove that π is not a fuzzy Hurewics fibration, suppose that μ_Y is fuzzy contractible. Also, if $\pi: \mu_X \rightarrow \mu_Y$ is fuzzy fibration, the admit a fuzzy section it would be necessary. In particular, let $H_t: \mu_Y \rightarrow \mu_Y$ be a fuzzy null-homotopy of id_{μ_Y} . Obviously, we can lift the constant map H_1 to μ_X . Suppose that π is fuzzy Hurewics fibration, the fuzzy lifting homotopy property then we get a lift $H^*: \mu_Y \rightarrow \mu_X$ and as a result a section H_0^* of π . But we showed previously that π admits no fuzzy section. Therefore π is not a fuzzy Hurewics fibration. Assume $\mu_G = \mu_{GL_{+1}}(\mathbb{R}) = (\mathbb{I}_+, +, \cdot)$ be the fuzzy topological abelian group which given by the positive reals with multiplication. Let the fuzzy trivial G-bundles over U and V , given by $\mu_{X_U} = U \times \mu_G$ and $\mu_{X_V} = V \times \mu_G$ respectively. The Fuzzy principal μ_G -bundle μ_X over μ_Y whose construct by gluing μ_{X_U} and μ_{X_V} along $U \cap V$ by the μ_G -isomorphism.

$$\varphi_f: \mu_{X_U}|_{U \cap V} \xrightarrow{\simeq} \mu_{X_V}|_{V \cap U} \quad (1)$$

Defined by

$$\varphi_f(x, g) = (x, f(x) \cdot g) \quad (2)$$

More concretely, μ_X is obtained from $\mu_{X_U} \amalg \mu_{X_V}$ by identifying $(x, g) \in \mu_{X_U}$ with $\varphi_f(x, g) \in \mu_{X_V}$ for each $x \in U \cap V$ for $x > 0$, the continuous function $f: U \cap V \rightarrow \mathbb{I}_+$ which does not extend to a continuous function $\mu_Y \rightarrow \mathbb{I}_+$. Where μ_Y be the space obtained by gluing to copies of $\mathbb{R}$ along $\mathbb{I}_+$

$$\mu_Y = (\mathbb{I} \times \{0,1\})/\mathbb{Z}_2$$

This space is not Hausdorff, let $q: \mathbb{I} \times \{0,1\} \rightarrow \mu_Y$ be the equation map. Define two subset covering μ_Y by $U = q(\mathbb{I} \times \{0\})$ and $V = q(\mathbb{I} \times \{1\})$.

The projection map $\pi: \mu_X \rightarrow \mu_Y$ gives a fuzzy principal μ_G -bundle over μ_Y , which comes with canonical isomorphism $\mu_{X_U} \simeq \mu_{X_U}$ and $\mu_{X_V} \simeq \mu_{X_V}$. We will now show that π does

not admit a section. Via the construction of μ_X , section of $\pi: \mu_X \rightarrow \mu_Y$ determine:

- A section of $\mu_{X_U} = U \times \mu_G \rightarrow U$, and therefore a map $s_U: U \rightarrow \mu_G = \mathbb{I}_+$; similarly, map $s_V: V \rightarrow \mu_G = \mathbb{I}_+$.
- These maps verify $s_V = f(x) \cdot s_U$ for each $x \in U \cap V$

In particular, $f(x) = s_{V(x)}/s_{U(x)}$ for all $x \in U \cap V$. Although, this implies that f extends to a continuous function $\bar{f}: X \rightarrow \mathbb{I}_+$ given by

$$\bar{f} = s_V(q(g(x), 1))/s_V(q(g(x), 0))$$

Which contradicts the known non-extension property of f

4. Fuzzy Slicing Structure Property

The slicing structure property introduced by [4] $p: E \rightarrow X$ if we given map. Via a slicing structure for p , the collection $S = \{\omega, \phi_U\}$ of the following entities:

- A system $\omega = \{U\}$ of open sets of X which covers X , called the slicing neighbourhoods.
- A system of maps $\{\phi_U/U \in \omega\}$ indexed by the slicing neighborhoods, called the slicing functions, where each ϕ_U , is defined on the subspace $U \times p^{-1}(U) \setminus \emptyset$, of the product space $X \times E$, with images in E in such a way that the following two conditions are satisfied

(SF1)

$$p\phi_U(a, e) = a \text{ where } a \in U, e \in p^{-1}(U)$$

(SF2)

$$\phi_U(p(e), e) = e \text{ (} e \in p^{-1}(U) \text{)}$$

If a slicing structure $S = \{\omega, \phi_U\}$ for p exists, we say that $p: E \rightarrow X$ has the slicing structure property (SSP).

$$BP \Rightarrow SSP \Rightarrow ParaCHP$$

If that $p: E \rightarrow X$ has the SSP then E is called sliced fiber space over X . In addition the triple (E, p, X) is called a *Hu* fibre space (for more details see) [8].

Definition 4.1 Let $\pi: \mu_X \rightarrow \mu_Y$. By a fuzzy slicing structure for π , we mean a collection $S = \{\omega, \phi_u\}$ of the following entities:

- A system $S = \{\omega, \phi_u\}$ of open sets of μ_Y which covers μ_Y , called the fuzzy slicing neighborhoods.
- A system of maps $\{\phi_u: u \in \omega\}$ indexed by the fuzzy slicing neighborhoods, called the fuzzy slicing function, where each ϕ_u , is defined on the subspace $u \times \pi^{-1}(u)$, respective of the product space $\mu_Y \times \mu_X$ respective with images in μ_X in such a way that the following two conditions are satisfied:

(SF1)

$$\pi \phi_u(y, x) = y, \text{ where } y \in u, x \in \pi^{-1}(u)$$

And

(SF2)

$$\phi_u(\pi(x), x) = x \text{ (} x \in \pi^{-1}(u) \text{)}$$

Definition 4.2 If a fuzzy slicing structure $S = \{\omega, \varepsilon, \phi_u\}$ for $\pi: \mu_X \rightarrow \mu_Y$ exists, we say that $\pi: \mu_X \rightarrow \mu_Y$, has the fuzzy slicing structure property (FuSSP). If $\pi: \mu_X \rightarrow \mu_Y$, has the MSSP then $\xi = (\mu_X, \pi, \mu_Y)$ is called a *Hu* fuzzy fibre space.

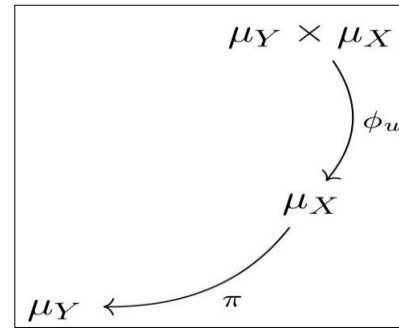


Fig 3: An illustration of the fuzzy principal μ G-bundle μ_X over μ_Y showing the construction by gluing $\mu_X U$ and $\mu_X V$ along $U \cap V \cup V \cap U$, with the μ G-isomorphism ϕ_f facilitating the identification of elements across the overlapping regions. This figure visually represents the non-triviality of the fuzzy principal bundle, demonstrating the failure to admit a section and the contradiction arising from the non-extension property of the function f .

Definition 4.3 A space μ_Y is fuzzy Locally Equi-Connected (FuLEC) if it has a cover by open sets V and for each V , a connecting maps $\phi: V \times V \rightarrow \mu_Y$ such that:

$$\phi(x, y)(0) = x, \sigma_1(x, y)(1) = y, \text{ and } \phi(x, x)(t) = x$$

Theorem 4.4 If μ_Y is fuzzy Locally Equi-Connected (FuLEC by symbol), then each regular fuzzy Serre fiber space (μ_X, π, μ_Y) is a *Hu* fuzzy fiber space.

Proof: Since (μ_X, π, μ_Y) is a regular fuzzy fibration, we have a regular lifting functions

$$\lambda: \Omega_\pi \rightarrow \mu_X$$

Where

$$\Omega_\pi = \{(x, \omega) \in \mu_X \times \mu_Y : \pi(x) = \omega(0)\}$$

And

$$\lambda(x, \omega)(0) = x, \pi \lambda(x, \omega) = \omega$$

And if ω, ε are constants, then $\lambda(x, \omega)$ is constant. Then we may define

$$\phi_V: U_V \times \pi^{-1}(U_V) \rightarrow \mu_X$$

By

$$\phi_V(y, x) = \lambda(x, \sigma_V(\pi(x)), y)(1),$$

Where $\{U_V\}$ cover for μ_Y and $\{\sigma_V\}$ is connecting map. Q_V a fuzzy slicing function so (μ_X, π, μ_Y) is a *Hu* fuzzy fibre space.

5. Conclusion

This study successfully introduced and explored the concept of Fuzzy Slicing Structure Property (FuSSP) within the context of fuzzy fiber spaces. By analyzing and proving that fuzzy fibrations do not necessarily qualify as fuzzy Hurewicz fibrations, we highlighted a significant distinction in fuzzy topological structures. The research provided key insights into the behaviors of fuzzy fiber bundles, paving the way for future studies in this area. The findings contribute to the broader understanding of fuzzy topology, offering a foundation for potential applications and further theoretical developments in this field.

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