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Effects of density and size on terminal velocity of a vertically falling spherical particles in Newtonian fluid by diagonal Pade' approximant

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Abstract

In this paper, the effects of physical parameters on terminal velocity of vertically falling spherical particles made of Glass, Iron, Copper and Silver with different diameters ($D=0.1\text{mm}$, 0.2mm , 0.5mm & 1mm) in Newtonian fluid is discussed using Diagonal Pade' [2/2] approximant and Collocation Method (CM) and compare the results with Runge-Kutta 4th order method to verify the accuracy. It observed that the Diagonal Pade' approximant which was used to solve nonlinear differential equations is more accurate and simpler as compared to Collocation Method (CM), Homotopy Perturbation Method (HPM), Akbari-Ganji's Method (AGM), and Variational Iteration Method (VIM) etc. The Outcomes clearly demonstrate that the time of reaching the particles at terminal velocity in a vertically falling procedure is significantly increased with growing the size and density of a particle and the acceleration period for smaller and lighter particles is shorter. Further from these four particles, the glass's particles have low velocity and reaches early at terminal velocity due to its lowest density as compared to other. To obtain the results for all different methods, the symbolic calculus software MATLAB was used.

Keywords: Collocation method (CM), diagonal pade' approximant, Newtonian fluid, spherical particle and terminal velocity

1. Introduction

The problem of motion of vertically falling spherical and non-spherical particles in Newtonian and Non-Newtonian fluids is relevant to many situations of practical interest. It is often essential to know the detailed trajectories of the accelerating particles in the fluid region for the purposes of designing or improving operation. For example, the distance required to reach the terminal velocity is necessary for the viscosity measurements of fluid using the falling ball method. It is also necessary to know the time and distance required to reach the particle at terminal point to determine the reliable results for design models. At the equilibrium stage, particle reaches a terminal velocity or also called settling velocity. Clift *et al.* and Chhabra^[33, 35] solved the problems of settling velocity of solid particle, drop, bubble in Newtonian and non-Newtonian fluids. Due to its wide use of applications in many industrial process, the motion of falling solid particles in liquids (fluids and gasses) has become a subject of grate interest e.g. Separation of liquid-solid mixtures, deposition in pipelines, alluvial channels, sediment transportation, powder processing, etc.^[12, 30]. L. Gmachowski^[17] obtained results from a tendency of suspended particles in fluids due to force acting on them, they settle and come to rest. The problems related to the motion of spheres and objects falling and rising in a fluids with both linear and quadric drag was solved by Mohazzabi and Guo^[11, 29]. Many researcher realized the physical importance of some analytical methods such as the Variational Iteration method (VIM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM) and its compatibility with problems as the unsteady motion of falling spherical particles in Newtonian fluids^[1-4]. Originally these methods were given by J.He^[15, 16] to achieve the series solution of strongly nonlinear differential equations. To study the unsteady motion of a spherical falling particles in Newtonian fluid for a range of Reynolds number to obtain a solution of nonlinear equation jalaal *et al.*^[13, 14, 25] used HPM. Hamidi *et al.*^[36] to develop the series solution of the couple equations of a spherical soiled particles motion

in plane Couette fluid flow also used HPM-Pade'. Hatami *et al.* [23] solved coupled equations of particle's motion in Couette fluid flow by Multi-step Differential Transformation Method (Ms-DTM) considering the rotation and shear effects on lift force and neglecting gravity. Also Hatami and Ganji [21] introduced the equation of motion of particles on a rotating parabolic surface through Lagrange equations and was solved by Ms-DTM with high degree of accuracy and least computational effort. It is clear from previous literature most of the investigations are performed for steady –state conditions where the particle achieved their terminal velocity and few of them has been studied the unsteady motion of falling particles. Hatami and Ganji [20] solved the couple equations of particle's motion in a fluid forced vortex using the differential transformation method (DTM) with the Padé approximation and the differential quadrature method (DQM). Ganji [4] derived a semi-exact solution for the instantaneous velocity of the particles over time in incompressible fluid by applied VIM. Yaghoobi and Torabi [6-7] also used DTM and VIM-Pade' for the solution of instantaneous velocity and acceleration motion of vertically falling non-spherical particles in Newtonian fluid and compared the results with VIM. Combined VIM to Pade' approximation for increasing the accuracy of results. Also Torabi and Yaghoobi [18] combined HPM with pade' approximation for increasing the accuracy of the equation of particle's motion and found that this method can achieve more accurate results. HPM also used by Jalaal *et al.* [24, 26] solution of problem related to motion of sphere rolling down an inclined plane submerged in Newtonian fluid. Hu [9] used Collocation Method for the solution of Poisson's equation with high accuracy. Herrera [10] also used Single collocation point method for the solution of advanced-diffusion equations. Majority of the literature review have describe the motion of solid particles in Newtonian fluid. In this paper, the terminal settling velocity and acceleration motion of spherical particles made of different materials (iron, copper, silver, glass) with different size, vertically falling in Newtonian fluid was considered. In terms of obtaining the best accuracy of CM and Diagonal Pade' method a compression was made by R-K 4th order method. From all these, one of the well-known analytical correlation between Reynolds numbers [8] and drag coefficient for sphere in a Newtonian fluid could be expressed as follows:

$$C_D = f(Re, n) \quad (1)$$

The drag coefficient could be obtained from Stokes law in following form:

$$C_D = \frac{24}{Re} X(n), \text{ Where } Re = \frac{\rho u^2}{\mu n D^{-n}} \text{ is the Reynolds number} \quad (2)$$

$$\text{and } X(n) = 6^{\frac{n-1}{2}} \left(\frac{3}{n^2+n+1} \right)^{n+1} \text{ Is a deviation factor} \quad (3)$$

Was obtained by many researcher with help of numerical or experimental result. From all these, one of the well-known analytical correlated equation of Renaud *et al.* [28] was used. In this work, we study terminal velocity and acceleration motion of vertically falling spherical particles made of iron, copper, silver, glass with diameters ($D=0.1\text{mm}$, 0.2mm , 0.5mm 1mm). The analysis derived by Diagonal Pade' approximant and CM ($i=2$). The results of current methods were compared with R-K 4th order method.

Nomenclature		Greek symbols	
α, β, γ	constants	μ	Dynamic viscosity, kg/ms
Acc	acceleration, m/s ²	ρ	Fluid density, kg/m ³
C_D	Drag coefficient	ρ_s	Spherical partical density, kg/m ³
D	Particle diameter, m		
g	acc. due to gravity, m/s ²		
m	particle mass, kg		
Re	Reynolds number		
t	time, s		
u	Velocity, m/s		

2. Problem Description

Consider a rigid body, spherical particle with equivalent volume diameter D , mass m and density ρ_s falling in an infinite extent of incompressible Newtonian fluid of density ρ and viscosity μ , u represents the velocity of the spherical particle at any instant time t , and g is the acceleration due to gravity [7]. Thus, the Basset -Boussinesq-Ossen (BBO) equation for the unsteady motion of particle in a fluid is given by [18]

$$m \frac{du}{dt} = mg \left(1 - \frac{\rho}{\rho_s} \right) - \frac{\pi D^2 \rho C_D}{8} u^n - \frac{\pi \rho D^3}{12} \frac{du}{dt} \quad (4)$$

Where C_D the drag coefficient. In right hand side of the Eq. (4), the 1st term represent the buoyancy effect, the 2nd term corresponds to drag resistance, 3rd term is associated with the added mass effect which is due to acc. of fluid around the particle. The complexity of the above equation arises due to the non-linear nature of drag coefficient. So by rewriting force balance Eq. (4) of motion of the particle, using Eq. (2) and Eq. (3) is

$$\alpha \frac{du}{dt} + \beta(n) u^n - \gamma = 0, u(0) = 0 \quad (5)$$

$$\text{In which } \alpha = m + \frac{1}{12} \pi D^3 \rho, \beta(n) = 3\pi K X(n) D^{2-n}, \gamma = mg \left(1 - \frac{\rho}{\rho_s} \right)$$

So for Newtonian fluid (n=1)

$$X(n) = 1,$$

$$\alpha \frac{du}{dt} + \beta u - \gamma = 0, u(0) = 0 \quad (6)$$

$$\alpha = m + \frac{1}{12} \pi D^3 \rho, \beta(1) = 3\pi D, \gamma = mg \left(1 - \frac{\rho}{\rho_s}\right) \quad (7)$$

3. Solutions of Problem

3.1 Diagonal Pade' [2/2] Approximants

A Pade' approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(t)$. Henri Pade' was developed this technique around 1890. The Padé approximant often gives better closed form approximation of the function and it may still work where the Taylor series does not converge. For these reasons Padé approximants are used extensively in computer calculation. They also have been used as in Diophantine approximation and transcendental number theory. The $[L/M]$ Pade' approximants to a function $u(t)$ are given by ^[23, 32]

$$\left[\frac{L}{M}\right] = \frac{p_0 + p_1 t^1 + p_2 t^2 + p_3 t^3 + \dots + p_L t^L}{q_0 + q_1 t^1 + q_2 t^2 + q_3 t^3 + \dots + q_M t^M}, \quad L=M \text{ (for diagonal pade' approximants)} \quad (8)$$

Take normalization condition $q_0=1$

$$u(t) - \frac{P_L(t)}{q_M(t)} = O(t^{L+M+1})$$

The formal power series is given

$$u(t) = \sum_{i=1}^{\infty} a_i t^i$$

i.e. $u(t) = a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + \dots$ (9)

Find the coefficients $a_0, a_1, a_2, a_3, \dots$ with help of Taylor's expansion. The given equation (6) becomes

$$u'(t) = \frac{\gamma}{\alpha} - \frac{\beta}{\alpha} u, \text{ with initial condition } u(0) = 0 \quad (10)$$

Solve the above equation by Taylor's series about zero is given by

$$u(t) = u_0 + t u'_0 + \frac{t^2}{2!} u''_0 + \frac{t^3}{3!} u'''_0 + \frac{t^4}{4!} u^{iv}_0 + \frac{t^5}{5!} u^v_0 + \dots$$

$$u(t) = 0 + t \frac{\gamma}{\alpha} + \frac{t^2}{2!} \left(-\frac{\beta\gamma}{\alpha^2}\right) + \frac{t^3}{3!} \left(\frac{\beta^2\gamma}{\alpha^3}\right) + \frac{t^4}{4!} \left(-\frac{\beta^3\gamma}{\alpha^4}\right) + \dots \quad (11)$$

From Eq. (9) and (11) $a_0 = 0, a_1 = \frac{\gamma}{\alpha}, a_2 = -\frac{\beta\gamma}{2\alpha^2}, a_3 = \frac{\beta^2\gamma}{6\alpha^3}, a_4 = -\frac{\beta^3\gamma}{24\alpha^4}$ and so on For Diagonal Pade' [2/2]

$$a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3 + \dots = \frac{p_0 + p_1 t^1 + p_2 t^2}{q_0 + q_1 t^1 + q_2 t^2} \quad (12)$$

$$\text{Generally, } p_L = a_L + \sum_{i=1}^{\min(L,M)} q_i a_{L-i} \quad (13)$$

$$\text{So diagonal Pade' [2/2]} = \frac{\frac{\gamma}{\alpha} t}{1 + \frac{\beta}{2\alpha} t + \frac{\beta^2}{12\alpha^2} t^2} \quad (14)$$

Table 1: Physical properties and values of constants of Eq. (6)

Particle	d(mm)	Density/(kg/m ³)	V=4/3πr ³	Mass(m)(gram)	α (kg/m ³)	β (kg/m ³)(1)	γ (kg/m ³)
glass	0.1	2590kg/m ³	5.2381x10 ⁴	1.3567x10 ⁻⁶	1.6178x10 ⁻⁹	9.4286x10 ⁻⁷	8.1768x10 ⁻⁹
	0.2		4.1905x10 ³	1.0853x10 ⁻⁵	1.2941x10 ⁻⁸	1.8857x10 ⁻⁶	6.5411x10 ⁻⁸
	0.5		6.5476x10 ²	1.6958x10 ⁻⁴	2.0221x10 ⁻⁷	4.7143x10 ⁻⁶	10.2205x10 ⁻⁷
	1.0		5.2381x10 ¹	1.3567x10 ⁻³	1.6178x10 ⁻⁶	9.4286x10 ⁻⁶	8.1768x10 ⁻⁶
Iron	0.1	7874kg/m ³	5.2381x10 ⁴	4.1245x10 ⁻⁶	4.3856x10 ⁻⁹	9.4285x10 ⁻⁷	3.5298x10 ⁻⁸
	0.2		4.1905x10 ³	3.2996x10 ⁻⁵	3.5084x10 ⁻⁸	1.8857x10 ⁻⁶	2.8239x10 ⁻⁷
	0.5		6.5476x10 ²	5.1556x10 ⁻⁴	5.4819x10 ⁻⁷	4.7143x10 ⁻⁶	4.4123x10 ⁻⁶
	1.0		5.2381x10 ¹	4.1245x10 ⁻³	4.3856x10 ⁻⁶	9.4286x10 ⁻⁶	3.5298x10 ⁻⁵
Copper	0.1	8940kg/m ³	5.2381x10 ⁴	4.6829x10 ⁻⁶	4.9440x10 ⁻⁹	9.4285x10 ⁻⁷	4.0770x10 ⁻⁸
	0.2		4.1905x10 ³	3.7463x10 ⁻⁵	3.9551x10 ⁻⁸	1.8857x10 ⁻⁶	3.2616x10 ⁻⁷
	0.5		6.5476x10 ²	5.8536x10 ⁻⁴	6.1799x10 ⁻⁷	4.7143x10 ⁻⁶	5.0963x10 ⁻⁶
	1.0		5.2381x10 ¹	4.6829x10 ⁻³	4.9440x10 ⁻⁶	9.4286x10 ⁻⁶	4.0770x10 ⁻⁵
Silver	0.1	10490kg/m ³	5.2381x10 ⁴	5.4948x10 ⁻⁶	5.7559x10 ⁻⁹	9.4285x10 ⁻⁷	4.8727x10 ⁻⁸
	0.2		4.1905x10 ³	4.3958x10 ⁻⁵	4.6046x10 ⁻⁸	1.8857x10 ⁻⁶	3.8982x10 ⁻⁷
	0.5		6.5476x10 ²	6.8685x10 ⁻⁴	7.1948x10 ⁻⁷	4.7143x10 ⁻⁶	6.0909x10 ⁻⁶
	1.0		5.2381x10 ¹	5.4948x10 ⁻³	5.7559x10 ⁻⁶	9.4286x10 ⁻⁶	4.8727x10 ⁻⁵

3.2 Collocation Method

To clarify the collocation method [34], suppose a differential operator D is acted on a function $u(t)$ to produce a function $p(t)$ is

$$D(u(t)) = p(t) \quad (15)$$

We want to approximate u by function $\hat{u}$, which is a linear combination of basic function chosen from a linearly independent set [27], i.e.

$$u \cong \hat{u} = \sum_{i=1}^n c_i \varphi_i \quad (16)$$

Now, when substituted $\hat{u}$ into differential operator D , the result of the operation is not $p(t)$. Hence an error and residual will exist as

$$E(t) = R(t) = D(\hat{u}(t)) - p(t) \neq 0$$

The main idea of the Collocation Method is to force the residual to zero in some average sense over the domain. i.e.

$$\int R(t) W_i(t) dt = 0, i=1, 2, 3 \dots n \quad (17)$$

Where the number of weight functions W_i is exactly equal to number of unknown constants c_i in $\hat{u}$ function. For the collocation method, the weighted function are taken from the family of Dirac δ function in the domain. i.e. $W_i(x) = \delta(x - x_i)$. The Dirac δ function has the property of [31, 34] $\delta(x - x_i) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$ and residual function must be forced to zero at specific point.

For solving Eq. (6) by WRMs, the trial function must satisfy initial condition in Eq. (6) i.e. $u(0) = 0$. So each statement in $u(t)$ should contain 't' to satisfy initial condition $u(0) = 0$. In this study statement is considered for velocity function which must satisfy initial condition and accuracy of results can be increased by increasing the number of statements (c_i), so

$$u(t) = c_1 t^1 + c_2 t^2 + c_3 t^3 + \dots + c_i t^i \quad (18)$$

Which satisfy the initial condition in Eq. (6) and by substituting Eq. (18) in Eq. (6) residual function $R(c_1, c_2, t)$ is found [34]. Take two specific points should be chosen in the domain t like t_1 and t_2 such that $R(c_1, c_2, t_1) = 0$ and $R(c_1, c_2, t_2) = 0$, a set of two equation with two unknown coefficients will be found. After solving these equations substitute constant coefficient into trial function, the velocity equation for particle will be determined as

$$u(t) = \frac{27\gamma}{27\alpha + 2\beta} t - \frac{9\beta\gamma}{27\alpha^2 + 2\alpha\beta} t^2 \quad (19)$$

3.3 Runge-kutta 4th order method

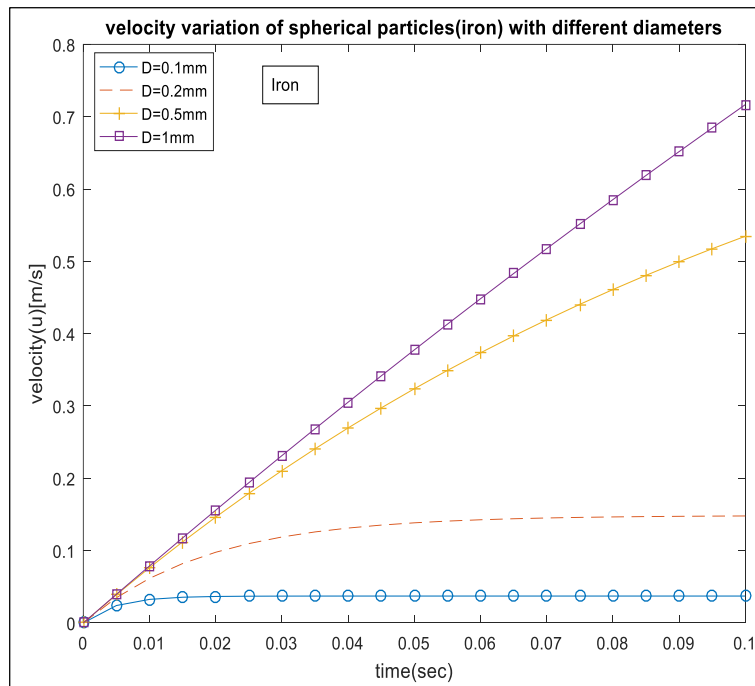
It is clear that the current problem is initial value problem (IVP) of 1st order. So far a solution, we can apply numerical methods like trapezoidal method, Euler's method (1st order R-K method), and R-K 4th order method. Trapezoidal method is generally used for typical problems. The R-K 4th order method is the modification in Euler's method by adding midpoint in the step which increase the accuracy. Thus R-K 4th order method is a suitable numerical technique in present problem R-K 4th order also known as Numerical Method (NM) [5, 27, 32].

In Table.1. The physical properties of particles made of different materials (made of glass, iron, copper, silver) and values of constant terms (α, β, γ) of equation (6) is illustrated. It is clear that the density of Silver's particles is highest and Glass's particles is lowest. Table 2. Shows the velocity values versus time of different particles with different diameters by R-K 4th order. Because of the highest density of silver particle, it has the highest velocity, whereas the glass particles, with lowest density, has the lowest velocity. In addition, velocity has a dramatic upward trends when the particle density and size growing because of increasing the mass of particles.

Table 2: Velocity of particles of different materials with different diameter by R-K 4th order Method

T (time)	Glass (u)			Iron (u)				Copper (u)				Silver (u)			
sec	0.2mm	0.5mm	1mm	0.1mm	0.2mm	0.5mm	1mm	0.1mm	0.2mm	0.5mm	1mm	0.1mm	0.2mm	0.5mm	1mm
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.005	0.0179	0.0239	0.0249	0.0243	0.0353	0.0394	0.0400	0.0263	0.0367	0.0405	0.0410	0.0288	0.0383	0.0416	0.0422
0.010	0.0266	0.0451	0.0491	0.0328	0.0623	0.0771	0.0796	0.0366	0.0655	0.0794	0.0817	0.0415	0.0695	0.0819	0.0840
0.015	0.0308	0.0640	0.0726	0.0358	0.0829	0.1133	0.1188	0.0407	0.0883	0.1169	0.1219	0.0472	0.0949	0.1209	0.1254
0.020	0.0328	0.0808	0.0954	0.0369	0.0986	0.1479	0.1576	0.0422	0.1062	0.1530	0.1618	0.0497	0.1156	0.1587	0.1666
0.025	0.0338	0.0958	0.1176	0.0372	0.1107	0.1811	0.1959	0.0428	0.1204	0.1877	0.2013	0.0508	0.1325	0.1952	0.2074
0.030	0.0342	0.1091	0.1391	0.0374	0.1199	0.2128	0.2338	0.0431	0.1315	0.2211	0.2404	0.0513	0.1462	0.2306	0.2478
0.035	0.0345	0.1209	0.1600	0.0374	0.1269	0.2433	0.2714	0.0432	0.1403	0.2533	0.2792	0.0515	0.1574	0.2648	0.2880
0.040	0.0346	0.1315	0.1803	0.0374	0.1323	0.2724	0.3085	0.0432	0.1472	0.2843	0.3176	0.0516	0.1665	0.2979	0.3278
0.045	0.0346	0.1409	0.2001	0.0374	0.1364	0.3003	0.3452	0.0432	0.1527	0.3141	0.3556	0.0516	0.1740	0.3299	0.3672
0.050	0.0347	0.1492	0.2192	0.0374	0.1396	0.3271	0.3816	0.0432	0.1570	0.3428	0.3933	0.0517	0.1800	0.3609	0.4064
0.055	0.0347	0.1567	0.2378	0.0374	0.1420	0.3527	0.4175	0.0432	0.1604	0.3704	0.4306	0.0517	0.1850	0.3909	0.4452
0.060	0.0347	0.1633	0.2559	0.0374	0.1438	0.3773	0.4531	0.0432	0.1630	0.3970	0.4675	0.0517	0.1890	0.4200	0.4838
0.065	0.0347	0.1692	0.2735	0.0374	0.1452	0.4008	0.4883	0.0432	0.1651	0.4226	0.5041	0.0517	0.1923	0.4481	0.5220
0.070	0.0347	0.1744	0.2905	0.0374	0.1463	0.4233	0.5231	0.0432	0.1668	0.4473	0.5404	0.0517	0.1950	0.4753	0.5599
0.075	0.0347	0.1791	0.3071	0.0374	0.1471	0.4449	0.5575	0.0432	0.1681	0.4710	0.5763	0.0517	0.1971	0.5016	0.5975
0.080	0.0347	0.1832	0.3232	0.0374	0.1477	0.4655	0.5916	0.0432	0.1691	0.4938	0.6118	0.0517	0.1989	0.5271	0.6347
0.085	0.0347	0.1869	0.3388	0.0374	0.1482	0.4853	0.6253	0.0432	0.1699	0.5158	0.6471	0.0517	0.2004	0.5517	0.6717
0.090	0.0347	0.1902	0.3540	0.0374	0.1486	0.5043	0.6586	0.0432	0.1706	0.5369	0.6820	0.0517	0.2015	0.5756	0.7084
0.095	0.0347	0.1931	0.3687	0.0374	0.1488	0.5225	0.6916	0.0432	0.1711	0.5573	0.7165	0.0517	0.2025	0.5987	0.7448
0.100	0.0347	0.1957	0.3830	0.0374	0.1491	0.5399	0.7242	0.0432	0.1715	0.5769	0.7508	0.0517	0.2033	0.6210	0.7809

Physical quantities of interest, the velocity versus time of particles made of different material with different size are shown in Fig. 1(a), (b), (c), (d). In fig 1(a) the iron particles has been taken with different size. In this, u (vertically) denotes the velocity of particles w.r.t. time t (horizontally) in seconds. Solution for velocity of the vertically falling spherical particles during the acceleration motion is obtain by R-k 4th order. All these figures shows that the particle's velocity is increasing as the particles size growing. The symbolic calculus software MATLAB is used.

**Fig 1(a):** Velocity variation of spherical particles (Iron) with different diameters in Newtonian Fluid by R-K 4th order method

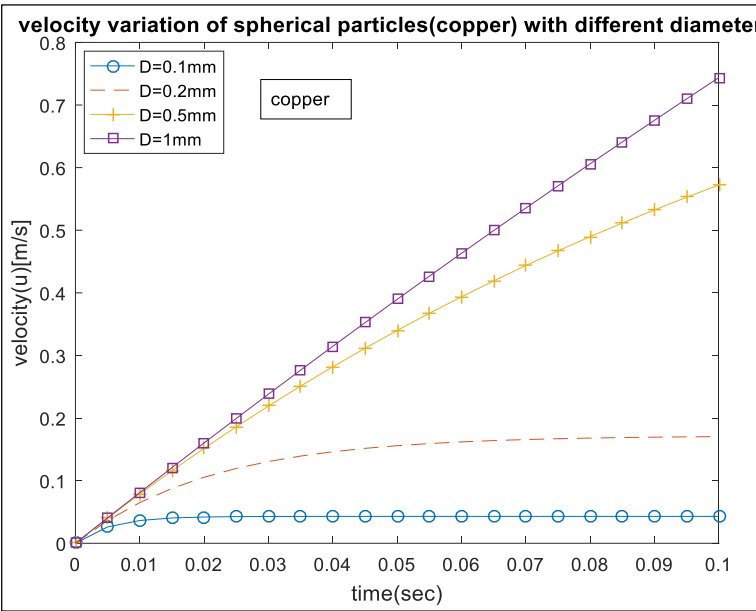


Fig 1(b): Velocity variation of spherical particles (Copper) with different diameters in Newtonian Fluid by R-K 4th order method

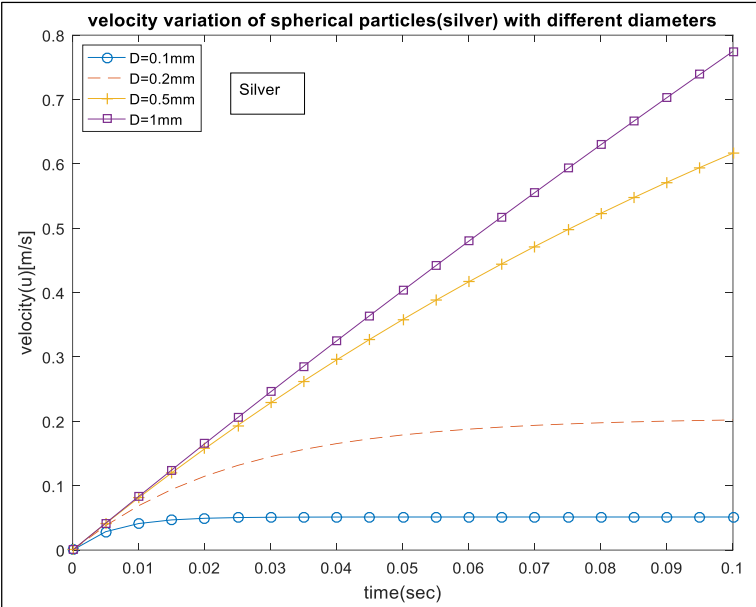


Fig 1(c): Velocity variation of spherical particles (Silver) with different diameters in Newtonian Fluid by R-K 4th order method

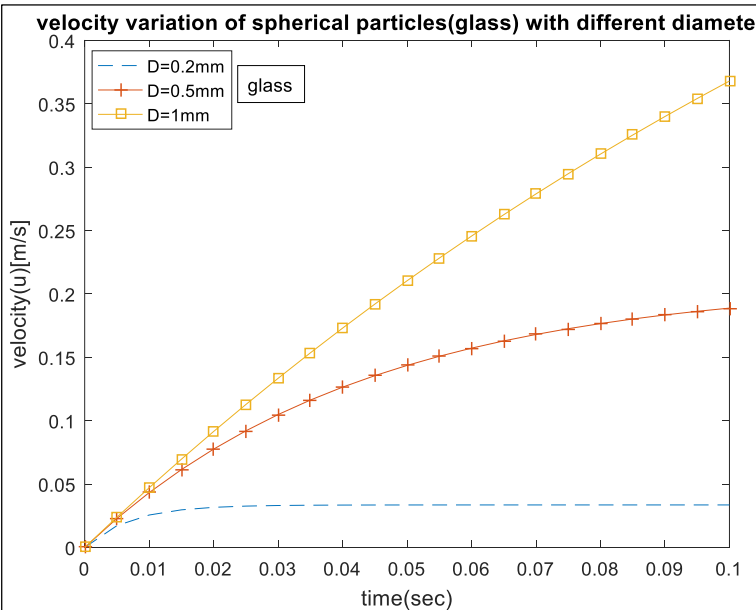


Fig 1(d): Velocity variation of spherical particles (Glass) with different diameters in Newtonian Fluid by R-K 4th order method

4. Results and Discussion

The applicability of the proposed methods for the non-linear equation of motion of settling particles will be discussed in this study. In order to measure the accuracy of the results, R-K 4th order method has been derived for nonlinear differential Eq. (6). The values of the fluid density (water) and consistency coefficient has been taken $\rho = 997 \text{ kg/m}^3$ and $K = 1$ respectively. The physical properties of particles and corresponding coefficient of Eq. (6) have been tabulated in Table.1. Terminal velocity of particles made of Iron, Copper, Silver and Glass with different size has been depicted in Table.3. It shows that due to increasing the size of particles, the terminal velocity also increasing. Same size particles made of different materials has been compared, the particles of glass has lowest terminal velocity and particles of Silver has highest terminal velocity. Matlab code was used to find the numerical solution of the present problem.

Table 3: Terminal Velocity [m/s] of particles made of different materials with different size

Particles	Terminal Velocity[m/s]			
	D=0.1mm	D=0.2mm	D=0.5mm	D=1mm
Glass	0.0087m/s	0.0347m/s	0.2168m/s	0.8672m/s
Iron	0.0374m/s	0.1498m/s	0.9359m/s	3.7437m/s
Copper	0.0432m/s	0.1730m/s	1.0747m/s	4.3241m/s
Silver	0.0516m/s	0.2067m/s	1.2920m/s	5.1680m/s

Settling velocity of different size particles made of different materials was shown in Figures 2(a), (b), (c), (d). It is obvious that the particles velocity is increasing until it reaches the terminal velocity. At that time particles settle and come to rest because the net force acting on them eliminates. Fig. 2(a) show that the effects of particle materials on the terminal velocity and time. Regarding Fig. 2(a), (b), (c) and (d), it is clear that the particles made of glass achieve terminal velocity in short time and has lowest velocity due to its lower density as compare to other.

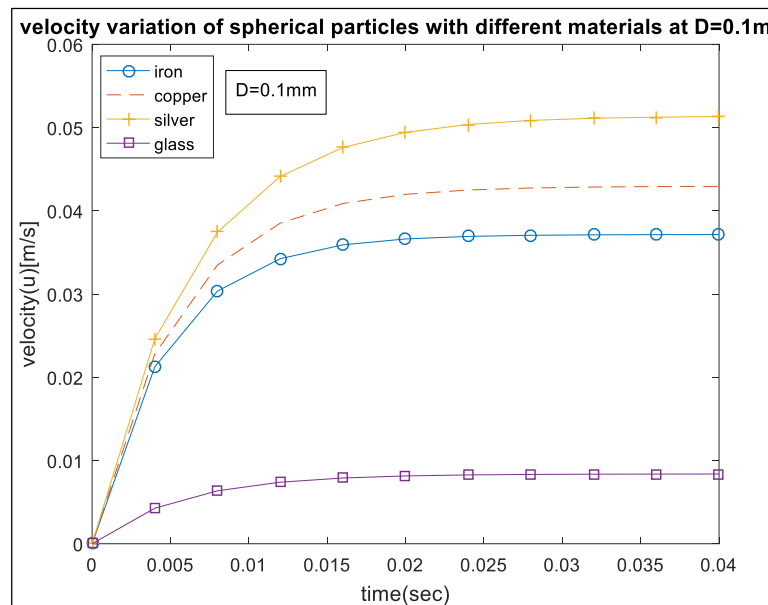


Fig 2(a): Terminal Velocity of spherical particles of different materials at D=0.1mm in Newtonian Fluid by R-K method

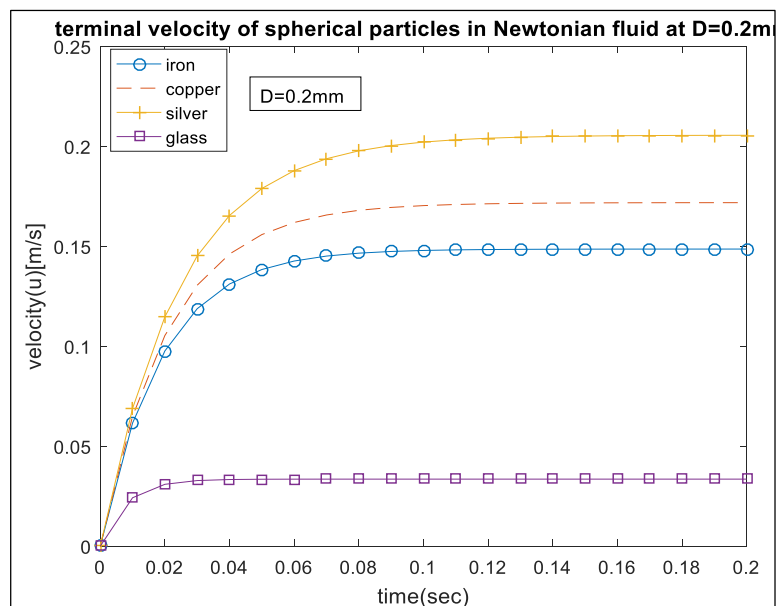


Fig 2(b): Terminal Velocity of spherical particles of different materials at D=0.2mm in Newtonian Fluid by R-K method

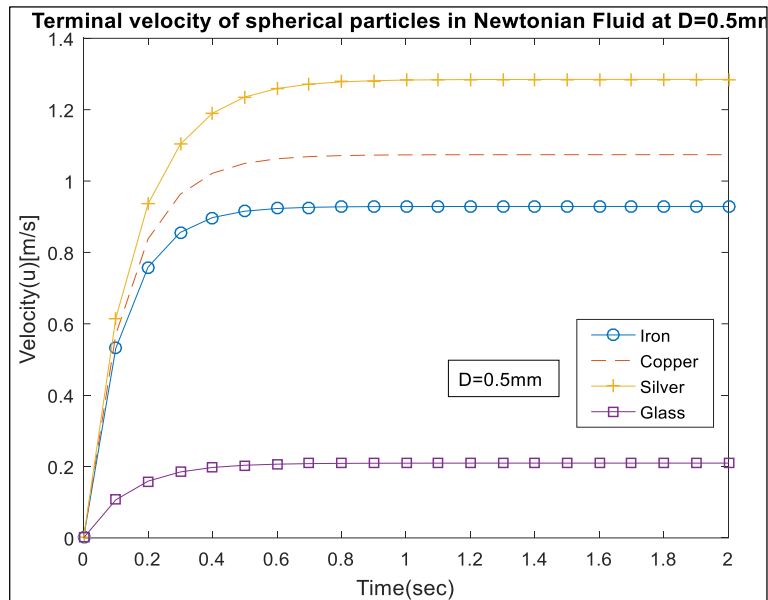


Fig 2(c): Terminal Velocity of spherical particles of different materials at D=0.5mm in Newtonian Fluid by R-K method

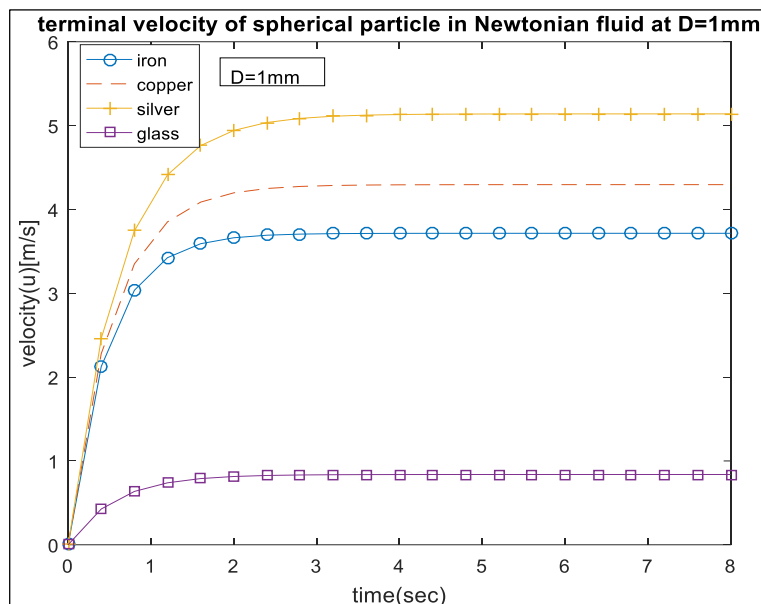


Fig 2(d): Terminal Velocity of spherical particles of different materials at D=1mm in Newtonian Fluid by R-K method

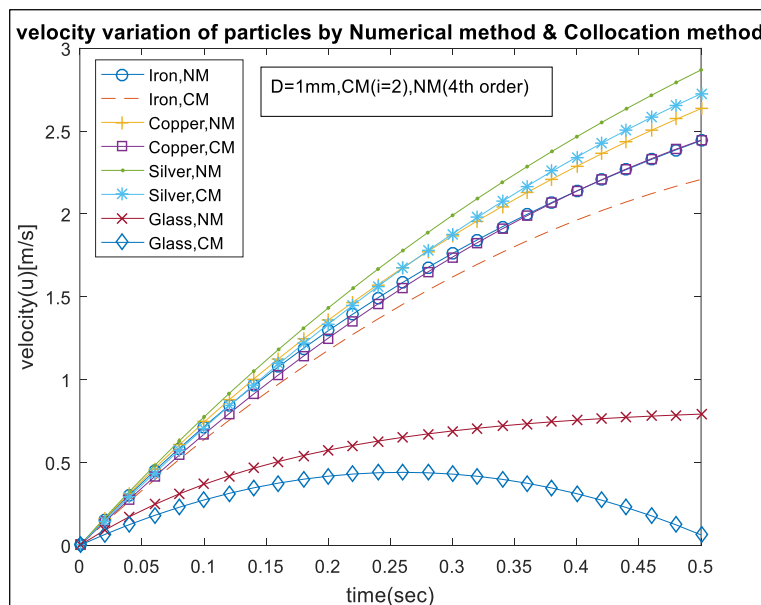


Fig 3(a): Velocity variation of spherical particles with different materials in Newtonian Fluid by R-K 4th order method (NM) and collocation method (CM)

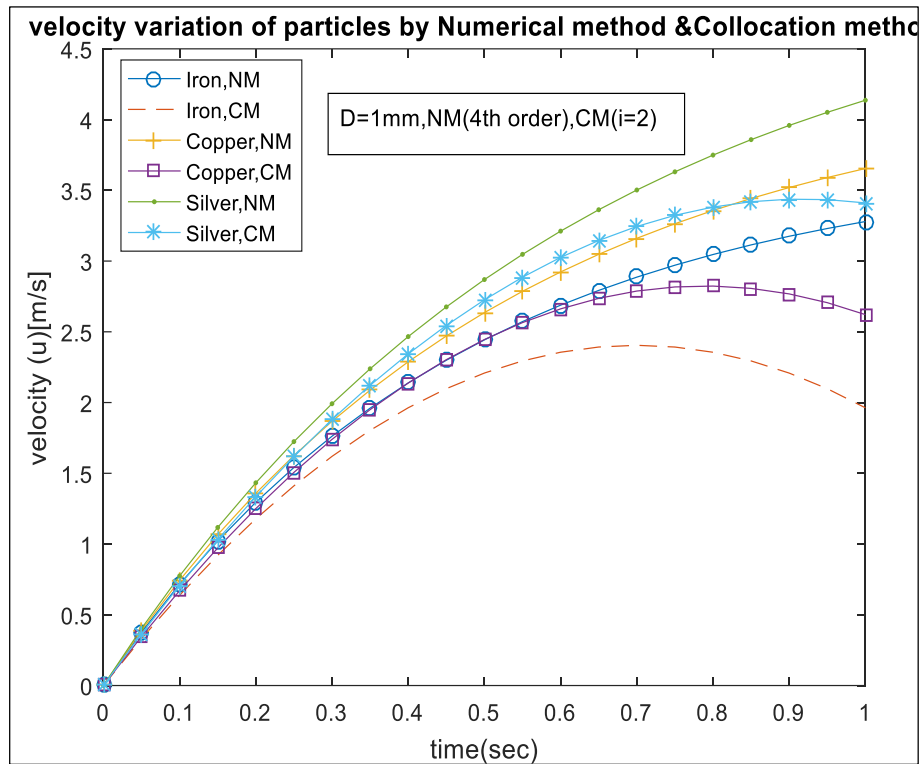


Fig 3(b): Velocity variation of spherical particles with different materials in Newtonian Fluid by R-K 4th order method (NM) and collocation method (CM)

Fig 3(a) and 3(b) shows the velocity results of same size ($D=1\text{mm}$) particles made of Iron, Copper, Silver and glass which obtained by Collocation Method ($i=2$) and R-k 4th order with diameter ($D=1\text{mm}$) in different time intervals. Clearly in short time collocation method gives accurate results. CM when time tends to infinity cannot estimate a terminal velocity and its value (Glass's particles) suddenly reaches to zero. Accuracy of CM will be increased by increasing the value of i in collocation method [34]. The velocity results of the present problem by Diagonal Pade' [2/2] are depicted in Fig. 4(a) and (b). From fig 4. It observes that Diagonal Pade' [2/2] has a good agreement and acceptable with numerical method.

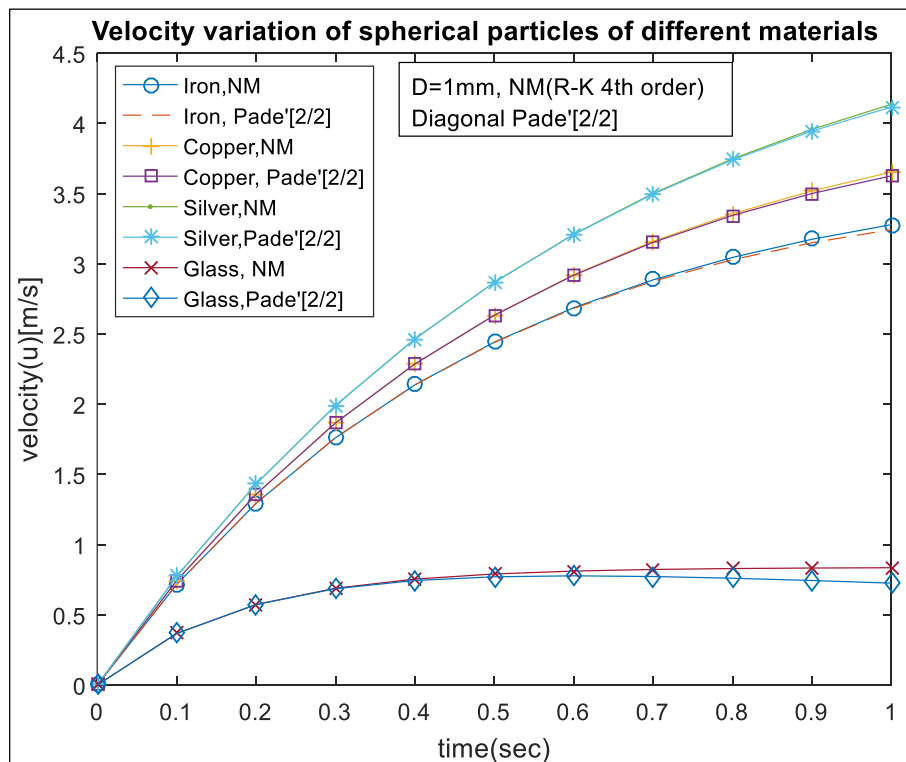


Fig 4(a): Comparison between R-K 4th order (NM) & Diagonal Pade' [2/2] of particles of different materials with diameter ($D=1\text{mm}$) in Newtonian fluid

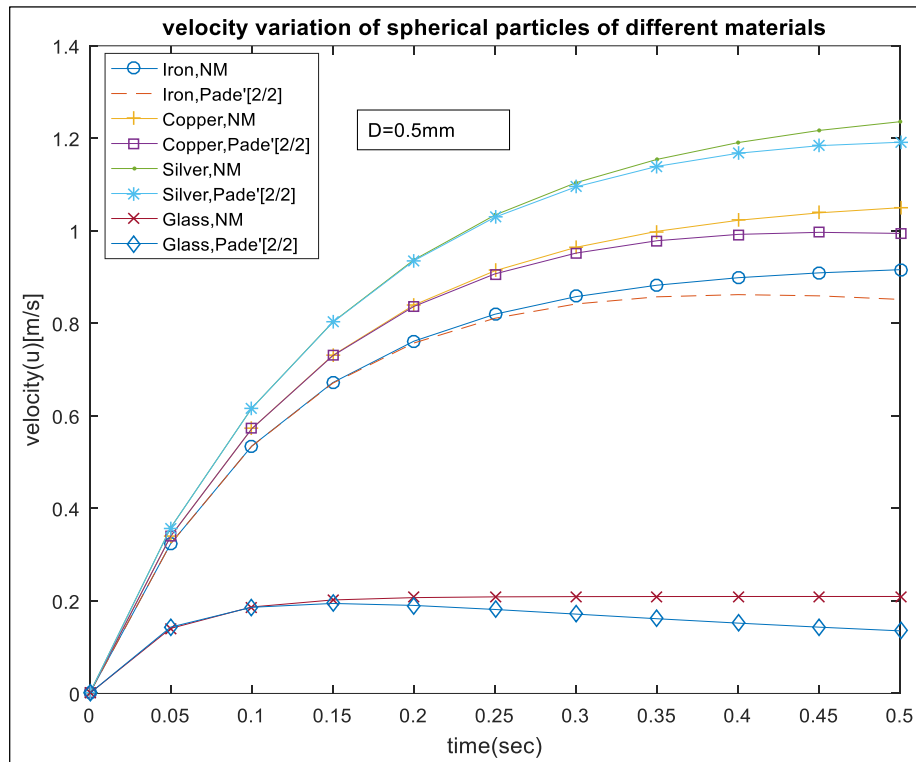


Fig 4(b): Comparison between R-K 4th order (NM) & Diagonal Pade' [2/2] of particles of different materials with diameter ($D=0.5\text{mm}$) in Newtonian fluid

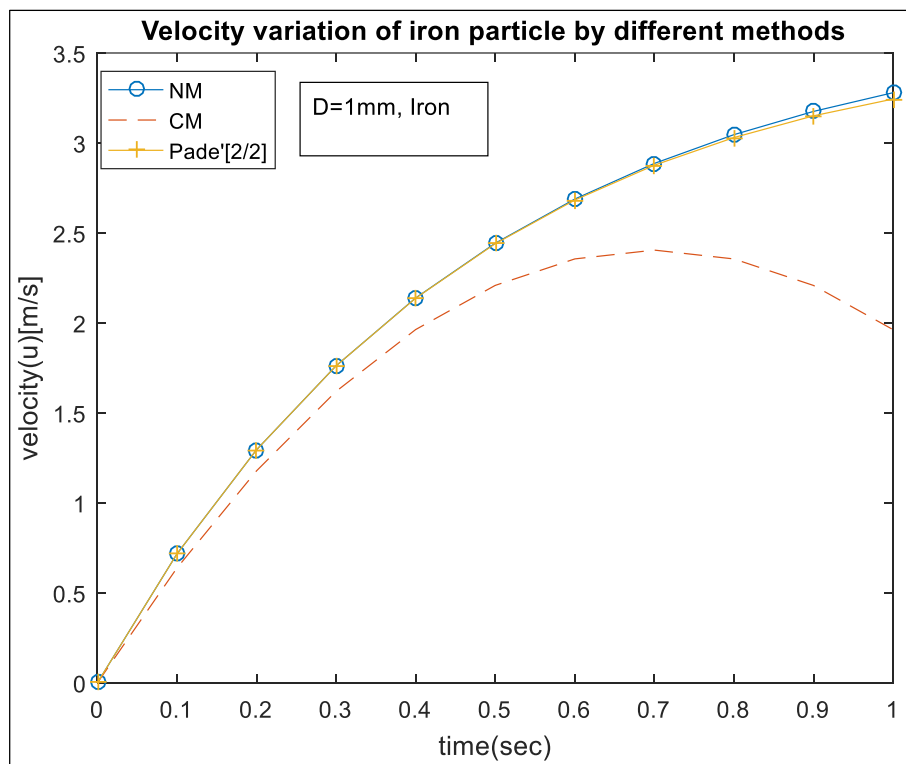


Fig 5(a): Comparison between R-K 4th order (NM), Collocation method and Diagonal Pade' [2/2] of particles of Iron with diameter ($D=1\text{mm}$) in Newtonian fluid

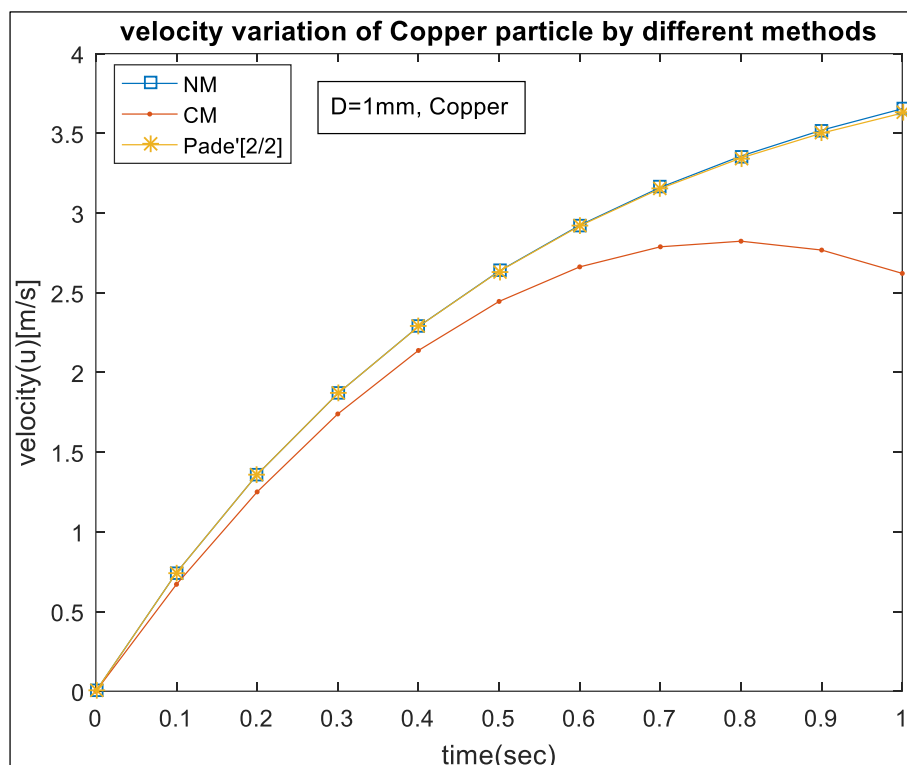


Fig 5(b): Comparison between R-K 4th order (NM), Collocation method and Diagonal Pade' [2/2] of particles of Copper with diameter (D=1mm) in Newtonian fluid

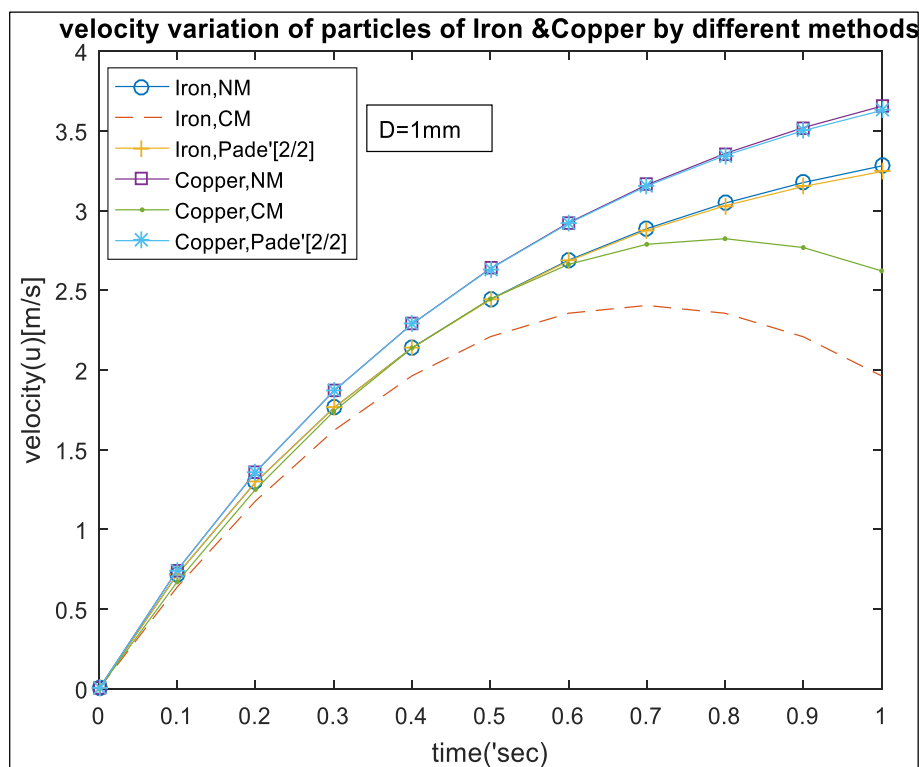


Fig 5(c): Comparison between R-K 4th order (NM), Collocation method and Diagonal Pade' [2/2] of particles of Iron and copper with diameter (D=1mm) in Newtonian fluid

Fig 5(a), (b), (c) shows the comparison of the velocity results of Diagonal Pade' approximant and CM with R-K method for particles of Iron and Copper. It was shown that Diagonal Pade' can lead into more accurate results compared to CM.

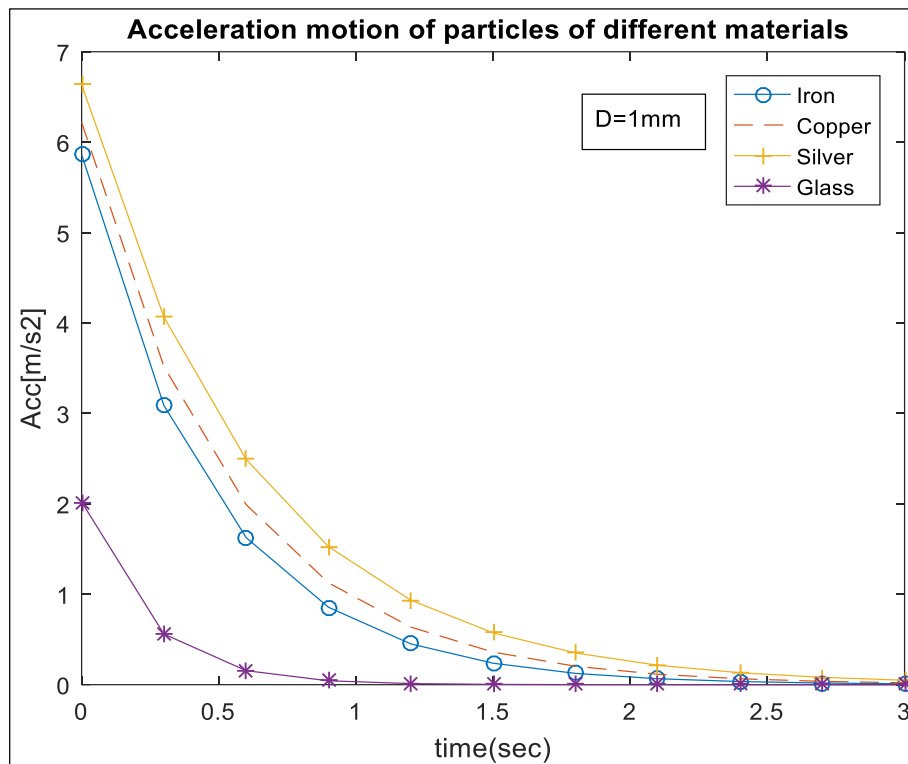


Fig 6(a): Acceleration motion of vertically falling spherical particles made of different materials ($D=1\text{mm}$) in Newtonian fluid

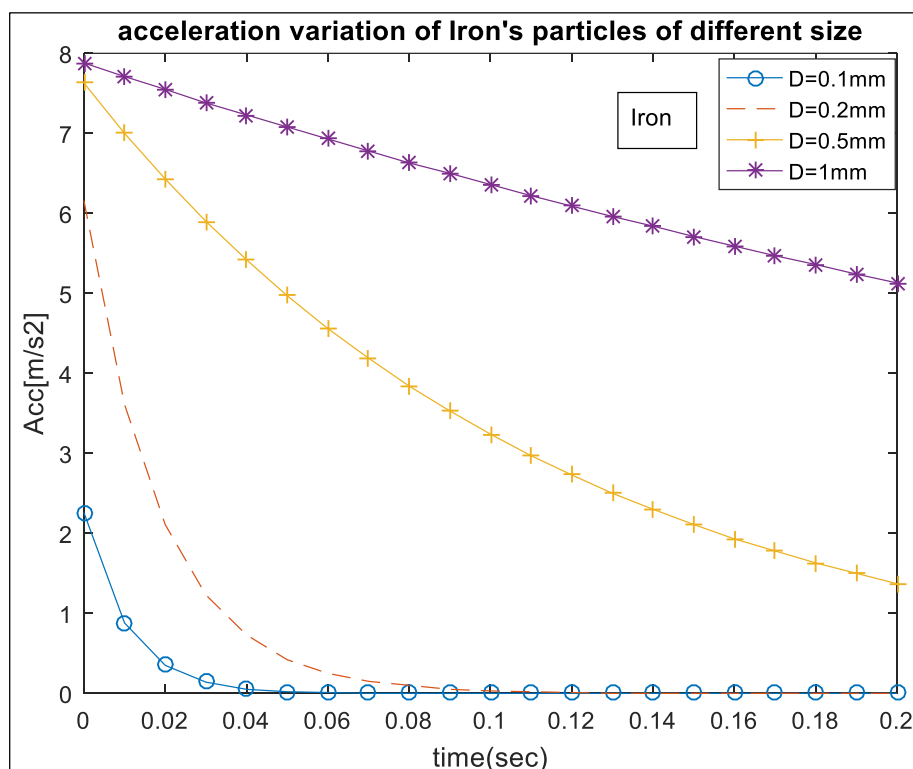


Fig 6(b): Acceleration motion of vertically falling spherical Iron's particles with different size in Newtonian fluid

In fig 6(a) and (b), it can be realized that the smaller particles to reaches to its terminal velocity (acceleration zero) earlier. The particles of lowest density has been taken less time to reaching terminal velocity as compare to other particles (i.e. acceleration motion of particles becomes zero). For a constant diameter, fig 6(a) indicates that initial acceleration time required to reach terminal velocity state increase by increasing the particle density. It also can be conclude that larger particles reaches zero acceleration more slowly.

5. Conclusion

The current methods are applied without using any linearization, discretization, restrictions or transformations. From above discussion, it is clear that the terminal velocity is increasing as growing the size of particles and the acceleration period for smaller and lighter particles are shorter. It also shows that the effectiveness and simplicity of the current mathematical methods. The Diagonal Pade' Approximant has a good agreement with R-k 4th order method and gives high degree of accuracy results. In

addition, this method does not require many calculation as CM to reach accurate results. Both methods gives the accurate results in short time, but Diagonal pade' method is also suitable for long time. Also, the current method (Diagonal Pade') can be used to develop the valid solution of other nonlinear differential equation of order one and more.

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Appendix-Matlab Code for R-K 4th order method % iron (D=1mm)

```
f = @(t,u) (gamma/alpha - beta/alpha * u)t = 0;

u = 0;

alpha = 4.3995 * 10^-6;

beta = 9.4286 * 10^-6;

gamma = 3.5030 * 10^-5;

h = 0.5;

t = 0:h:3;

for i = 1:(length(t) - 1);

k1 = f(t(i),u(i));

k2 = f(t(i) + 0.5 * h,u(i) + 0.5 * h * k1);

k3 = f(t(i) + 0.5 * h,u(i) + 0.5 * h * k2);

k4 = f(t(i) + h,u(i) + h * k3);

u(i + 1) = u(i) + 1/6 * (k1 + 2 * k2 + 2 * k3 + k4) * h;

end u(:)
```

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