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Investigate future population projection of Bangladesh with the help of Malthusian model, Sharpe-lotka model and Gurtin Mac-Camy model

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Abstract

A nonlinear deterministic model of age-dependent population growth is formulated in terms of joint integral equations in the total population size function and the total birth rate function. Existence of a unique solution of the model equation is established using Banach fixed point theorem. Then we investigate the total population size of Malthusian model, Sharpe-Lotka linear model and Gurtin Mac-Camy model for some specific age group and compare our investigated solutions with the total population of Bangladesh by using numerical method "Newton divided difference formula". In our present study, we also investigate future population projection of Bangladesh with the help of Malthusian model, Sharpe-Lotka model and Gurtin Mac-Camy model.

Keywords: Nonlinear, population model, age-time population, Malthusian model, Sharpe-Lotka linear model, Gurtin Mac-Camy model

1. Introduction

The study of age-dependent population models originated with the work of Sharpe and Lotka [3], in which the birth rate is expressed as the solution of the renewal equation, a linear Volterra integral equation. Nonlinear age-dependent models may be constructed beginning with the MacKendric equation, and the basic model that we shall use is the one formulated by Gurtin-MacCamy [7, 8] in which the birth and death moduli may depend on age and total population size. In our present paper, population of Bangladesh has been predicted with the help of an ordinary differential equation model known as Malthusian model which is parameterized by growth rate along with specific age group and also predicted with the help of linear continuous Sharpe-Lotka population model and nonlinear Gurtin Mac-Camy model. We use "Newton's divided difference formula" scheme for the numerical solution. First we consider different types of data table using [9] which are depend on the specific age groups and then we find our population projection which is very good fit with the actual data. After that we also investigate population projection of Bangladesh with the help of Malthusian model, Sharpe-Lotka and Gurtin Mac-Camy model using same data table. In our present study we also investigate the future population projection of Bangladesh with respect to Malthusian model, Sharpe-Lotka model and Gurtin Mac-Camy model by using references [10, 11].

2. Method Formulation

2.1 Malthusian Model

The dynamics of a single species population can be described by

$$P'(t) = P(t)f(t, P(t))$$

where, $P(t)$ = size of population at time t .

The simplest assumption is that of constant growth rate which depends only on birth and death processes. i.e. $f(t, P(t)) = b - d$ = intrinsic growth rate of population

where, b = intrinsic birth rate d = intrinsic death rate

Now if we consider that $f(t, P(t))$ is a constant function, then

$$P'(t) = aP(t)$$

where, $f(t, P(t)) = a = \text{constant}$

This is a linear differential equation and is known as the Malthusian law of population growth.

2.2 Age-Structured Sharpe-Lotka Model

Let $x(t, a)$ be the population density (or age-distribution) with respect to age a at time t .

Let, $\nu(a)$ = an age dependent birth rate, $\mu(a)$ = an age dependent death rate.

$\mu(a)x(t, a)dt$ = number of the population of age a that dies in the small increment of time dt .

$x(t, a)\Delta a$ = number of individuals with age between a and $a + \Delta a$ at time t .

$$\sum x(t, a)\Delta a = \text{total population at time } t.$$

$$\sum x(t, a)\Delta a \rightarrow \int_0^{\infty} x(t, a)da \quad \text{as } \Delta a \rightarrow 0$$

$$\sum x(t, a)\Delta a \rightarrow \int_0^{\infty} x(t, a)da \quad \text{as } \Delta a \rightarrow 0$$

$\therefore x(t) = \int_0^{\infty} x(t, a)da$ is the total population. Note that, $x(t, a) = 0$ for all sufficiently large a , so that the above integration is not

necessarily an infinite integral. We assume that members leave the population only through death and that there is an age dependent death rate $\mu(a)$. $\mu(a)\Delta t$ denote the fraction of the individuals die over the interval from t to $t + \Delta t$ with ages between a and $a + \Delta t$ at time t . So $x(t, a)\Delta a\mu(a)\Delta t$ denote the number of deaths between the time t and $t + \Delta t$ with ages between a and $a + \Delta a$ and the remaining ages between $a + \Delta t$ and $a + \Delta t + \Delta a$ at time Δt .

$$\therefore x(t + \Delta t, a + \Delta t)\Delta a \approx x(t, a)\Delta a - x(t, a)\mu(a)\Delta a\Delta t \quad (2.2.1)$$

Dividing (2.2.1) by $\Delta a\Delta t$, we get

$$\begin{aligned} \frac{x(t + \Delta t, a + \Delta t)\Delta a}{\Delta a\Delta t} - \frac{x(t, a)\Delta a}{\Delta a\Delta t} + \frac{x(t, a)\mu(a)\Delta a\Delta t}{\Delta a\Delta t} &= 0 \Rightarrow \frac{x(t + \Delta t, a + \Delta t)}{\Delta t} - \frac{x(t, a)}{\Delta t} + x(t, a)\mu(a) = 0 \\ \Rightarrow \frac{x(t + \Delta t, a + \Delta t) - x(t, a)}{\Delta t} + x(t, a)\mu(a) &= 0 \end{aligned} \quad (2.2.2)$$

Let $\Delta t \rightarrow 0$ and $x(t, a)$ is a differentiable function of a and t , then we have

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t, a + \Delta t) - x(t, a)}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t, a + \Delta t) - x(t + \Delta t, a)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t, a) - x(t, a)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} x_a(t + \Delta t, a) + x_t(t, a) \\ &= x_a(t, a) + x_t(t, a) \\ &= \frac{\partial x(t, a)}{\partial a} + \frac{\partial x(t, a)}{\partial t} \\ &= D_e x(t, a) \end{aligned}$$

where $D_e x(t, a)$ is a directional derivative of x .

$$\therefore D_e x(t, a) = \frac{\partial x}{\partial a} + \frac{\partial x}{\partial t}$$

Now from (2.2.2) we have

$$\begin{aligned} D_e x(t, a) + x(t, a)\mu(a) &= 0 \\ \Rightarrow x_a(t, a) + x_t(t, a) + x(t, a)\mu(a) &= 0 \end{aligned} \quad (2.2.3)$$

This is called the Von Foerster equation and more rapidly called the McKendrick equation and Equation (2.3.3 which condition on $x(t, a)$ in t and a .

2.3 Gurtin-MacCamy Model

The problem is based on the following initial-boundary value problem:

$$\frac{\partial x(a, t)}{\partial a} + \frac{\partial x(a, t)}{\partial t} + \mu(a, P(t))x(a, t) = 0, \quad a > 0, t > 0$$

$$x(0, t) = \int_0^{\infty} \nu(a, P(t))x(a, t)da, \quad t \geq 0$$

$$x(a, 0) = \varphi(a), \quad a \geq 0$$

$$P(t) = \int_0^{\infty} x(a, t)da, \quad t \geq 0$$

In this model we describe for the dynamics of a biological population, closed to migratory movements, involves two independent variables a and t ($a \geq 0, -\infty < t < \infty$) which representing age and time respectively measured in the same unit and the following non-negative functions: (i) the total population size function $P(t)$ (ii) the total birth rate function $B(t)$ (iii) the age-density function $x(a, t)$ (iv) the natality function $\nu(a, t, P)$, $P \geq 0$ (v) the mortality function $\mu(a, t, P)$, $P \geq 0$ (vi) the initial age distribution function $\varphi(a)$ Regarding these functions, the following observations are in order. (1) $x(a, t)$ is the population of age a at time t . Hence the total population at time t is

$$P(t) = \int_0^{\infty} x(a, t)da \quad (2.3.1)$$

Also the number of new-borns at time t is, $B(t) = x(0, t)$, (2) $\nu(a, t, P)$ is the average number of children born to an individual of age a at time t when the total population is P and $\nu(a, t, P(t))x(a, t)dad t$ is the number of offspring produced between the times t and $t + dt$ by individuals with ages between a and $a + da$. It follows that

$$B(t) = \int_0^{\infty} \nu(a, t, P(t))x(a, t)da, \quad t > 0 \quad (2.3.2)$$

(3) $\mu(a, t, P)$ is the death rate of the population of age a at time t when the total population is P . Thus $\mu(a, t, P(t))x(a, t)dad t$ is the number of individuals with ages between a and $a + da$ who die between the times t and $t + dt$. (4) For any infinitesimal age-increment da , the number of individuals which are in the age-category $[a + da]$ at time t is equal to $x(a, t)da$. In a small time increment Δt , the age of each individual is increased by Δt Consequently,

$x(a + \Delta t, t + \Delta t)da - x(a, t)da$, represents the net loss of individuals in the age-category $[a, a + da]$ during the time interval $[t, t + \Delta t]$. Since, there is no migration, this loss is entirely due to death and thus

$$[x(a + \Delta t, t + \Delta t) - x(a, t)]da = -\mu(a, t, P(t))x(a, t)da\Delta t \text{ or,}$$

$$[x(a + \Delta t, t + \Delta t) - x(a, t) + \mu(a, t, P(t))x(a, t)\Delta t]da = 0 \text{ Dividing both sides by } da\Delta t,$$

$$\text{or, } [x(a + \Delta t, t + \Delta t) - x(a, t) + \mu(a, t, P(t))x(a, t)\Delta t]da = 0 \text{ Dividing both sides by } da\Delta t,$$

$$\frac{x(a + \Delta t, t + \Delta t)}{\Delta t} - \frac{x(a, t)}{\Delta t} + \mu(a, t, P(t))x(a, t) = 0$$

$$\text{or, } \frac{x(a + \Delta t, t + \Delta t) - x(a, t)}{\Delta t} + \mu(a, t, P(t))x(a, t) = 0$$

$$\text{or, } (*)$$

Let $\Delta t \rightarrow 0$ and $x(a, t)$ is differentiable function of a and t , then we have

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{x(a + \Delta t, t + \Delta t) - x(a, t)}{\Delta t} &= \lim_{\Delta t \rightarrow 0} \frac{x(a + \Delta t, t + \Delta t) - x(a, t + \Delta t)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{x(a, t + \Delta t) - x(a, t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} x_a(a, t + \Delta t) - x_t(a, t) \end{aligned}$$

$$= x_a(a, t) - x_t(a, t) \\ = \frac{\partial x(a, t)}{\partial a} + \frac{\partial x(a, t)}{\partial t}$$

$Dx(a, t) = \lim_{\Delta t \rightarrow 0} \frac{x(a + \Delta t, t + \Delta t) - x(a, t)}{\Delta t}$ exists. So from equation (*), $Dx(a, t) = -\mu(a, t, P(t))x(a, t)$, Let $S(a, t; P)$ denote the probability that an individual born at time will survive to age a . Then out of $x(0, t)$ infants born at time t , $x(0, t)S(a, t; P)$ individuals survive to age a and $x(0, t)S(a + \Delta a, t; P)$ individuals survive to age $a + \Delta a$. The difference of these two numbers is the fraction of the number of individuals born at time t that survive to age a and die between the ages a and $a + \Delta a$, i.e. between the times $t + a$ and $t + a + \Delta a$. Thus

$$x(0, t)[S(a, t; P)S(a + \Delta a, t; P)] = -x(0, t)S(a, t; P)\mu(a, t + a, P(t + a))\Delta a$$

$$\text{or, } \frac{S(a, t; P) - S(a + \Delta a, t; P)}{\Delta a} = -S(a, t; P)\mu(a, t + a, P(t + a))$$

Taking limit as $\Delta a \rightarrow 0$ on L.H.S we have,

$$\lim_{\Delta a \rightarrow 0} \frac{S(a, t; P) - S(a + \Delta a, t; P)}{\Delta a} = -S(a, t; P)\mu(a, t + a, P(t + a))$$

or, $S_a(a, t; P) = -\mu(a, t + a, P(t + a))S(a, t; P)$ where, $S_a = \frac{\partial S}{\partial a}$, Since $S(0, t; P) = 1$, this y

$$S(a, t; P) = \exp\left\{-\int_t^{t+a} \mu(\tau - t, \tau, P(\tau))d\tau\right\}$$

under the assumption that $P(t)$ and $\mu(a, t, P)$ are continuous functions of their arguments. Clearly $S > 0$ and is monotonic decreasing with respect to a . Now, let $\Pi(a, t; P)$ denote the probability that an individual (born at time $t - a$) will survive to age a at time t . Then $\Pi(a, t; P) = S(a, t - a; P) = \exp\left\{-\int_{t-a}^t \mu(\tau - t + a, \tau, P(\tau))d\tau\right\}$ Also for $a \geq t$, let $\lambda(a, t; P)$ be the probability that an individual who has age $a - t$ at time 0 will survive to age a at time t . Then $\lambda(a, t; P)$ is the conditional probability of survival to age a at time t , given the survival to age $a - t$ at time 0 and hence Let us now obtain the integral equations relating $P(t)$ and $B(t)$. Obviously, $x(a, t)$ is the fraction of the number of individuals born at time $t - a$ that survive to age a at time t . Hence, for $t > a$, Also, for $a \geq t \geq 0$, $x(a, t)$ equals the fraction of the number of individuals with age $a - t$ at time 0 that survive to age a at time t and hence

$$x(a, t) = \varphi(a - t)\lambda(a, t; P), \text{ Thus we have, } x(a, t) = \begin{cases} B(t - a)\Pi(a, t; p), & a < t \\ \varphi(a - t)\lambda(a, t; P), & a \geq t \end{cases}$$

From (2.3.1) and (2.3.2), we have

$$P(t) = \int_0^t B(t - a)\Pi(a, t; P) da + \int_t^\infty \varphi(a - t)\lambda(a, t; P) da \\ = \int_0^t \Lambda(a, t; P) B(a) da + \int_0^\infty \Gamma(a, t; P)\varphi(a) da \quad (2.3.3)$$

$$B(t) = \int_0^t v(a, t; P(t)) B(t - a)\Pi(a, t; P) da + \int_t^\infty v(a, t; P(t)) \varphi(a - t) \lambda(a, t; P) \varphi(a - t) da \\ = \int_0^t v(t - a, t; P(t)) \Lambda(a, t; P) B(a) da + \int_0^\infty v(t + a, t; P(t)) \Gamma(a, t; P) \varphi(a) da \quad (2.3.4)$$

where $\Lambda(a, t; P) = \Pi(t - a, t; P) = \exp\left\{-\int_a^t \mu(\tau - a, \tau, P(\tau))d\tau\right\}$

and $\Gamma(a, t; P) = \lambda(t + a, t; P) = \exp\left\{-\int_0^t \mu(\tau + a, \tau, P(\tau))d\tau\right\}$

3. Discussion about Findings&Graphical Analysis

3.1 Relationship of Proposed Methods with the Population of Bangladesh

In this chapter we work with some population data which we found from the Population Census. We put these data in the above discussed model and verify these with Bangladesh population.

Table 1: Women aged 15-39 years and over by age group and children died after birth, 2011.

Age	Number of Children	Children died after birth								
		Total	%	Bangladesh	Malthusian model	Sharpe-Lotka model	Gurtin-MacCamy model	Error of Malthusian	Error of Sharpe-Lotka	Error of Gurtin-MacCamy
15-19	6108871	34758	.578	0.82	2.82	0.883	1.269	0.085	0.02	0.02
20-24	6613990	307515	4.75	18.6	34.28	22.968	27.539	1.133	0.739	.897
25-29	6957607	767016	11.13	154.39	221.45	166.118	184.563	7.363	5.455	6.01
30-34	5107978	911809	17.96	640.63	974.68	779.773	835.46	33.00	26.380	28.20
35-39	5072599	1290484	25.55	1830.56	3575.74	2207.757	3303.788	122.53	75.35	113.49

From the above table we see that in Bangladesh prospect Malthusian, Sharpe-Lotka, Gurtin-MacCamy none of these model is consistent. If we put population of Bangladesh in these model they rapidly change their behavior and population value vary. To find the error or the rate of change of population here we use the numerical method (Newton's Divided Difference Formula). The formula is,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + \dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})f(x_0, x_1, \dots, x_{n-1})$$

Figure-1: Shows the significant effect of Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model with Bangladesh population curve. Population curves do not coincide with each other. Malthusian $P(t)$ curve is more exponential than Gurtin-MacCamy. Gurtin-MacCamy and Sharpe-Lotka $P(t)$ curves are more exponential than Bangladesh $P(t)$ curve. So Gurtin-MacCamy and Sharpe-Lotka models are little bit closure to the population increasing process of Bangladesh.

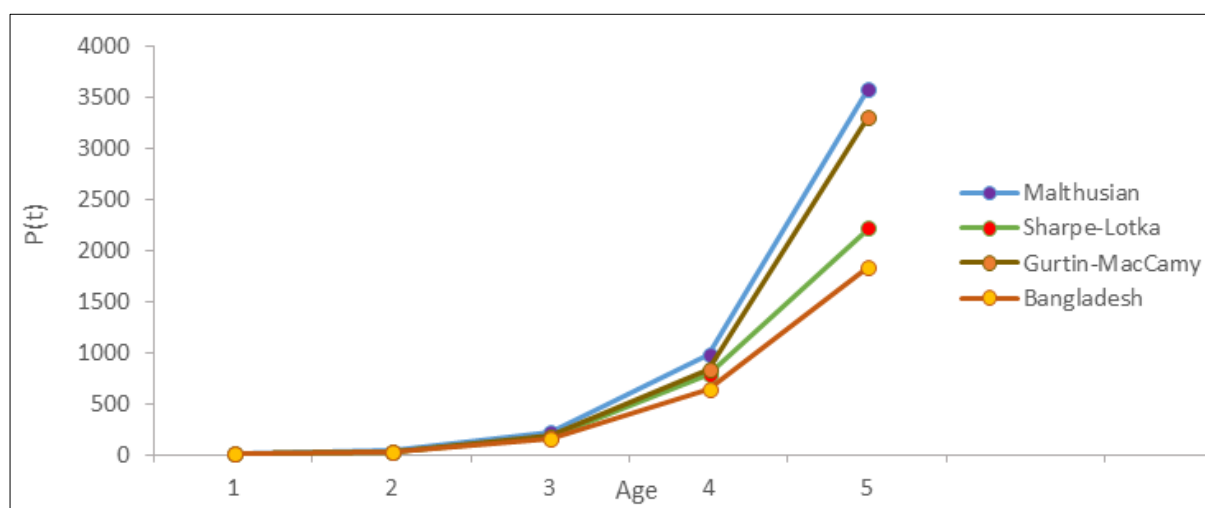


Fig 1: Women aged 15-39 years and over by age group and children died after birth, 2011.

Table 2: Population aged 5-34 years and over by group deaths in the last 12 months by residence, age and sex of the deceased, 2011.

Age	Total	Bangladesh	Malthusian model	Sharpe-Lotka model	Gurtin-MacCamy model	Error of Malthusian	Error of Sharpe-Lotka	Error of Gurtin-MacCamy
5-9	1342584	0.09	0.09	0.219	0.274	0.01	0.130	0.019
10-14	1134660	2.116	4.6	3.126	3.716	0.05	0.03	0.04
15-19	876231	15.176	24.2	18.178	20.197	0.3	0.27	0.29
20-24	739194	35.136	59.62	47.710	49.614	0.6	0.57	0.59
25-29	719029	85.134	154.4	128.736	135.167	1.6	1.3	1.4
30-34	566816	215.56	476.15	408.140	423.263	4.3	4.3	4.3

From the above table we see that in Bangladesh prospect Malthusian, Sharpe-Lotka and Gurtin-MacCamy none of these model is consistent. If we put population of Bangladesh in these model they rapidly change their behavior and population value vary. To find the error or the rate of change of population here we use the numerical method (Newton's Divided Difference Formula). The formula is,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + \dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})f(x_0, x_1, \dots, x_{n-1})$$

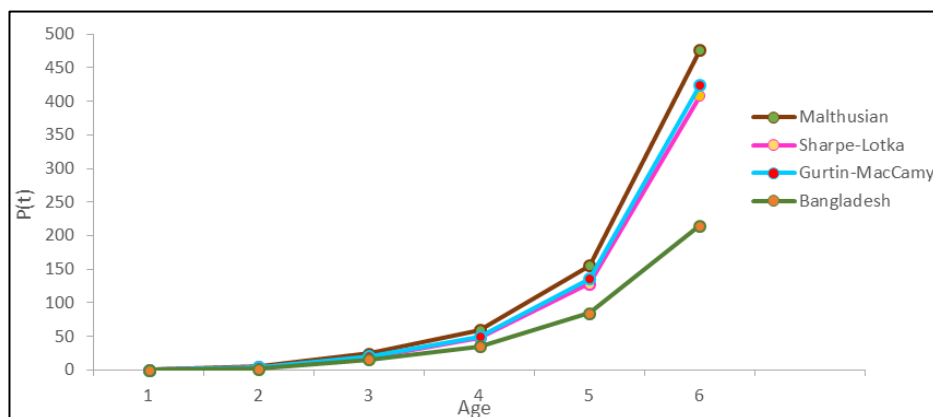


Fig 2: Population aged 5-34 years and over by age group deaths in the last 12 months by residence, age and sex of the deceased, 2011.

Table 3: Maternal deaths 12 months prior to the survey by age and residence, 2011.

Age	Total women	Total maternal deaths	Per capita maternal deaths in Bangladesh	Malthusian model	Sharpe-Lotka model	Gurtin-MacCamy model	Error of Malthusian	Error of Sharpe-Lotka	Error of Gurtin-MacCamy
10-14	7734517	297	0.0048	0.020	0.0061	0.0088	0.007	0.0002	0.0005
15-19	6108871	1075	0.0186	0.130	0.0872	0.1245	0.02	0.009	0.009
20-24	6613990	1115	0.0279	0.450	0.2563	0.2912	0.04	0.03	0.04
25-29	6957607	1068	0.0164	0.940	0.6784	0.727	0.09	0.07	0.08
30-34	5107978	480	0.0104	1.490	1.1613	1.219	0.14	0.2	0.2
35-39	5072599	1210	0.0349	9.400	8.358	8.613	0.97	0.9	0.9

From the above table we see that in Bangladesh prospect Malthusian, Sharpe-Lotka and Gurtin-MacCamy none of these model is consistent. If we put population of Bangladesh in these model they rapidly change their behavior and population value vary. To find the error or the rate of change of population here we use the numerical method (Newton's Divided Difference Formula). The formula is,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + \dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})f(x_0, x_1, \dots, x_{n-1})$$

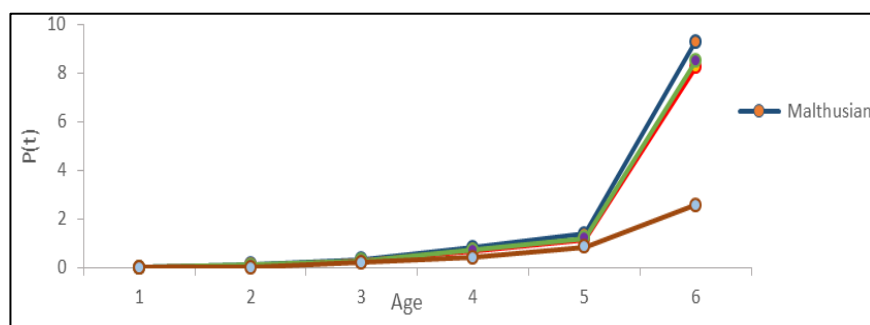


Fig 3: Maternal deaths 12 months prior to the survey by age and residence, 2011.

3.2 Future Population Projection of Bangladesh

Table 4: Population of Bangladesh in 2021 with respect to the Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model

Year		Population(Crores)		
	Bangladesh	Gurtin-MacCamy model	Sharpe-Lotka model	Malthusian model
1981	9.8	5.7	6.3	8.3
1991	11.9	9.3	18.3	22.4
2001	13.6	31.4	52.3	60.4
2011	15.3	98.5	148.6	163.9
2021	17.05	296.9	420.1	445.3

From the above table we see that in Bangladesh prospect Malthusian, Sharpe-Lotka, Gurtin-MacCamy none of these model is consistent. If we put population of Bangladesh in these model they rapidly change their behavior and population value vary exponentially. Their differences are too high. But comparatively Gurtin-MacCamy model is little bit closer to the Bangladesh population. From the following graph we see that, population of Bangladesh is increasing day by day. Following figure indicates that population growth of Bangladesh is not exponential as like Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model. So Malthusian model, Sharpe-Lotka model, Gurtin-MacCamy model is not applicable to determine the future population of Bangladesh. But if we want to compare graphically Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model, we say that Malthusian curve is more exponential than Bangladesh curve, Sharpe-Lotka and Gurtin-MacCamy curves are also exponential but Gurtin-macCamy curve is little bit closure to the Bangladesh curve.

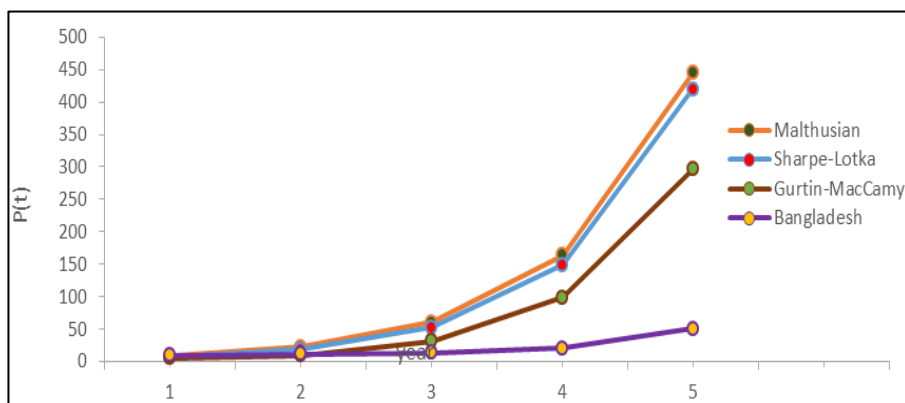


Fig 4: Population of Bangladesh in 2021 with respect to Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model.

3.3. Comparisons

The comparisons of future population of Bangladesh are presented in Table-05. This table shows good agreement between our present study and reference ^[12]. In our paper, we predict future population of Bangladesh by “Least Square Method”.

Table 5: Future Population of Bangladesh

Year	Present Study	Ref ^[12]
1981	9.8	9.05
1991	11.9	11.34
2001	13.6	12.96
2011	15.3	14.87
2021	17.05	17.25

4. Conclusion

In this paper, we have worked with Malthusian model, Verhulst Logistic Model, Sharpe-Lotka model and Gurtin Mac-Camy model. Here Malthusian, Verhulst and Sharpe-Lotka model are linear and Gurtin Mac-Camy model is non-linear. The description of the Gurtin Mac-Camy model constitute with the integral equations in the population size function and the birth rate function. Existence of a unique solution of the model equation is established using Banach fixed point theorem. We also examine the steady state and the stability of the model. In our present study we investigate the total population size of Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model for some specific age group. Then we compare our investigated solutions with the total population of Bangladesh for same age group. After comparisons we conclude that Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model shows more exponential result and these three models are not appropriate for Bangladesh. But from graphical representation we see that the Gurtin-MacCamy curve is little bit closure to the Bangladesh curve. We also determine the future population projection of Bangladesh with the help of Malthusian model, Sharpe-Lotka model and Gurtin-MacCamy model.

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